The Beijer Institute of Ecological Economics

DISCUSSION PAPER

Beijer Discussion Paper Series No. 259

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Ram Fishman and Chandra Kiran Krishnamurthy

The Beijer Institute of Ecological Economics
The Royal Swedish Academy of Sciences
Box 50005 SE-104 05 Stockholm Sweden
Phone: +46 8 673 95 00 Fax: +46 8 15 24 64
E-mail: beijer@beijer.kva.se

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Ram Fishman†, Chandra Kiran Krishnamurthy‡,§

November 7, 2017

Abstract

Most renewable biotic resources are subject to random variability in natural growth. We investigate the implications of such variability for long-term management by a risk averse social planner, who maximizes expected long-run utility. In the canonical model of a stochastic fishery, we show that the optimal level of harvesting effort need not necessarily be reduced by the introduction of variability in stock growth. However, optimal effort is reduced if the variability of growth increases for smaller base populations, as suggested in the ecology literature.

JEL classification: D90, D63, Q20, Q22, C62

Keywords: Stochastic Growth, Natural Resource, Golden Rule

1 Introduction

The natural growth of many biotic resources and ecological populations is subject to substantial random fluctuations, either as a result of natural variation or stochastic environmental conditions. Natural variability in stock levels can translate into variability in harvests, particularly if harvesting effort levels are not adjusted rapidly enough to adapt to short term fluctuations, potentially leading to loss of welfare. It is sometimes suggested in the ecology literature that stochastic variability in a species' growth implies

*We are thankful to Geoff Heal for comments that helped substantially improve the quality of our discussion, and to John Stachurski for helpful suggestions. The usual disclaimer applies.

†Department of Economics, George Washington University and the Department of Public Policy, Tel Aviv University. Email:rfishman@gwu.edu

‡Beijer Institute of Ecological Economics, The Royal Swedish Academy of Science, Stockholm. email: chandra.kiran@beijer.kva.se

§Department of Forest Economics, Swedish University of Agricultural Sciences, Umeå.

the level of harvesting effort should be reduced (for example, Roughgarden and Smith (1996)). Our analysis examines the question in the canonical model of a stochastic fishery, in which the welfare costs of variable harvests are captured by a concave utility function. Our main result in this setting is that variability in stock growth need not necessarily reduce the optimal level of effort. We show that the crucial characteristic of resource growth shocks is not uncertainty per se but rather, whether uncertainty is stock-dependent in a particular way: variability is higher at lower stock levels. It is only under this condition that the optimal effort is reduced in comparison with the deterministic model.

We note that increasing variability of growth at lower stock levels has been emphasised as being an important attribute of many ecological systems (see e.g. Lande (1993)). One reason for demographic variability to increase at low population levels is that the group average of the independent chances of individual mortality and reproduction growth rates fluctuate around the mean, with a variance that scales inversely with the size of the population. Another reason is that smaller populations are likely to be more vulnerable to changes in environmental conditions (Leizarowitz and Tsur (2012)).

The literature in economics on the management of stochastic natural resources has focused on the implication of stochastic growth for the nature of the inter-temporally optimal policy, i.e. the harvesting policy that maximises the discounted utility of harvest, and the possibility of extinction (Reed, 1975, 1978, 1979; Beddington and May, 1977; Getz et al., 1987; Costello et al., 2001; Sethi et al., 2005; Mitra and Roy, 2006; McGough et al., 2009; Leizarowitz and Tsur, 2012; Olson and Roy, 2000). This paper is concerned instead with a long-term perspective, and thus studies a different welfare criterion, the long-run level of the expected utility of harvest. The problem is a natural extension of the *golden rule* of saving (Phelps, 1961) and the more recently introduced *green golden rule* of resource extraction (Chichilnisky et al., 1995) to a stochastic framework.

In this context, we examine the impact of introducing uncertainty in stock growth for choice of the optimal level of harvesting effort. While there is no possibility of extinction in our model, (steady state) variability in catch is nonetheless welfare-reducing.[1] In this dynamic economic model, it is rigorously established that choice of the 'best' steady state need not necessarily be affected by stochastic growth. But if the variance in growth rate increases for smaller base populations, as suggested in the ecology literature, we find that the golden rule level of effort is *reduced* in comparison to the deterministic case. In fact, the stock-dependent uncertainty we consider may be viewed as altering the extraction-saving trade-off in a transparent manner, leading to reduced extraction. Consequently, this optimal policy leads to a "precautionary stock accumulation" effect, wherein stationary stocks are increasing in the degree of stock-dependence of uncertainty. The model used is specific, with functional forms, in particular, chosen to allow closed form solutions, but the basic insights are likely to hold in broad generality, and their formal generalization is left for future work.

The paper is organized as follows. In section 2, the general resource extraction problem is set up, with the deterministic version discussed in section 3. Section 4 considers this problem under stock-dependent uncertainty, establishes a result analogous to the 'green golden rule' of Chichilnisky et al. (1995) for effort level, and provides a brief discussion. Section 5 concludes. All technical details are collected in the Appendices.

2 Resource Dynamics

To illustrate our ideas, we introduce stochastic growth factors into the canonical model of fishery dynamics used in the seminal work of Levhari and Mirman (1980). Let the size of the fish population at time t be given by X_t. The natural dynamics of the fishery are governed by the equation

$$X_{t+1} = g_t X_t^{\alpha}, \tag{1}$$

where $0 < \alpha < 1$ is a fixed exponent and g_t are stochastic (random variables) growth factors, whose distribution $F_X(.)$ can depend upon the stock level X, but which we

[1] Aspects of harvesting policies, particularly for fisheries, in randomly fluctuating environments where the growth rates have certain specific characteristics has been explored before in the ecological literature (see e.g. Beddington and May (1977), Parma (1990)). While related, the question we consider is different, and so too the methods we use.

otherwise assume to be stationary.

The social planner chooses a level of harvest effort, $e \in (0,1)$, which describes the number of boats, the time spent fishing, etc.. The level of effort embodies a choice that is hard to change over time and is represented in the model by a constant.[2] We follow standard models by assuming that the catch (fish consumption) is proportional to both the (regenerated or 'grown') stock level and the level of effort, i.e.

$$C_{t+1} = e g_t X_t^{\alpha} \tag{2a}$$

$$X_{t+1} = (1-e) g_t X_t^{\alpha}. \tag{2b}$$

Finally, the utility of consumption is assumed to be logarithmic,

$$U_t = \log(C_t). \tag{2c}$$

We note that the timing of uncertainty is conventional: every period t, the planner chooses a (time-invariant) effort level, e, following which the uncertain growth shocks are received, leading to the next period stock.

We use the standard set-up of stochastic economic dynamic frameworks (e.g. Stachurski (2009); Hopenhayn and Prescott (1992)), with a key difference: stock, X_t, is drawn from a set, $\mathcal{X} \subset \mathbb{R}_+^{\setminus\{0\}}$, which is not necessarily compact. We define the ecological golden rule as the level of effort, e, which maximizes asymptotic expected utility,

$$\widehat{e} := \underset{e \in (0,1)}{\arg\max} \left\{ \mathbb{E}\left(U^{\star}(e)\right) \right\}. \tag{3}$$

This formulation is intended to extend the setting of Chichilnisky et al. (1995) to a stochastic context. In other words, our objective function is

$$\max_{e \in (0,1)} \lim_{t \to \infty} \mathbb{E}\Big[U\left(C(X_t, e)\right) \Big], \tag{4}$$

[2] The assumption of constant effort is quite common (see e.g. Beddington and May (1977) and references therein) and can be justified as being not far from optimal (Parma (1990)).

with C and X as in eq. (2).

As regards technical preliminaries, being time-invariant, effort e parameterizes the stock transitions. Furthermore, for every choice of effort e, the stock dynamics in eq. (2b) represents a Markov chain on state space $\mathcal{X}$. Thus, for the problem in eq. (4) to be well-defined the Markov chain $\{X_t, t \geq 0\}$ must converge to an invariant distribution; associated functions of this chain, U, C, must also converge. These details are discussed in section 4. We note, however, that establishing convergence of the Markov chain, and in particular of its functions, is a challenging endeavour when the shocks, g, are stock-dependent, as in our case.[3]

3 The Deterministic Case

We begin by revisiting the familiar deterministic case and then introduce stochastic growth. In the deterministic case $g_t \equiv 1$ so that natural growth is described by

$$X_{t+1} = X_t^{\alpha}. \tag{5}$$

Steady states stock levels are given by $X = 0$ (unstable) and $X = 1$ (stable). With harvesting effort e, the system dynamics are simply

$$\begin{aligned}
C_{t+1} &= eX_t^{\alpha}, \\
X_{t+1} &= (1-e)X_t^{\alpha}, \\
U_{t+1} &= \log(C_{t+1}) = \log(e) + \alpha \log(X_t).
\end{aligned}$$

In this model, every effort level stabilizes the system in a finite steady state, with

[3]The framework outlined here, and developed in section 4.1 and section 4.2, is a stochastic version of the the solow growth model. Our analysis builds upon prior investigations of aspects of this model in the macro-economics literature (e.g. Stachurski (2003), Van and Stachurski (2007)).

the following stock and utility:

$$X^\star = (1-e)^{\frac{1}{1-\alpha}} \tag{6a}$$

$$U^\star = \log(e) + \frac{\alpha}{1-\alpha}\log(1-e). \tag{6b}$$

The *golden rule* level of effort strives to balance the positive effect of effort on harvest (consumption) with its negative effect on stock. It is easy to see that this balance is struck at

$$e^\star = 1-\alpha. \tag{7}$$

This is the deterministic version of the problem in eq. (3). Two simple points regarding this set up are worth mentioning. First, in the deterministic case, maximizing the steady state level of utility of consumption is equivalent to maximising the steady state level of consumption itself, so the utility function plays no special role . This will no longer be true in the stochastic case. Second, we have not introduced a cost to exerting effort into the model. Nonetheless, even when effort is costless, it is not optimal to set effort to its maximal level, $e = 1$. Increasing effort increases consumption given a stock level, but it also reduces the level of the stock in steady state. The balance between these two effects is struck by the golden rule, and is completely analogous to the golden rule level of saving in the Solow growth model.

4 The Stochastic Fishery

We now incorporate stochastic growth and return to the full model described in eq. (2). In the interest of focusing on intuition, we begin first with a tractable framework, using a specialized form for the growth factors. Having established the ecological golden rule with this form, we turn to a more general framework for constructing distributions meeting the requirements of our application, and establish a few important results that will make rigorous the arguments leading to our main result. Finally, we also show that steady state stock, when extraction effort is optimally chosen, is increasing in the degree of stock dependence.

4.1 The Basic Model

We first develop a probabilistic framework for the growth shocks in which lower stock levels lead to increased variance, as often discussed in the ecological literature. Similarly, in the spirit of the relevant literature, the focus of our analysis is on "small-but-positive" level of stocks, with stock dependence of the form we consider of little interest at large levels of stock. In particular, we will assume that g_t is conditionally log-normally distributed

$$g_t|X_t \sim \log \mathcal{N}(\mu(X_t), \sigma^2(X_t)), \tag{8a}$$

and impose the specific functional form for its conditional moments:

$$\mathbb{E}(g_t|X_t) = 1, \tag{8b}$$

$$\mathbb{V}(g_t|X_t) = vX_t^{-2\lambda} - 1. \tag{8c}$$

In eq. (8b), we assume, for convenience and without loss of generality, that the growth factors all average to 1 while in eq. (8c), we impose a specific form on the variance of the growth distribution. Here, $v - 1$ is the variance of g_t when $X_t = 1$ (we assume that $v \geq 1$),[4] and $\lambda \geq 0$ is a parameter that describes how strongly variability in growth rates declines with stock levels. When $\lambda = 0$, the variance (and the entire distribution of g) is independent of the level of the stock. The larger the value of λ, the more rapid the increase in variance at lower levels of the stock. To satisfy the moments detailed in eq. (8), the scale of the log normal distribution will need to be set to $\sigma^2(X_t) = \log(v) - 2\lambda \log(X_t)$.[5]

[4] The conditions under which this particular parameterization is sensible are detailed in Remark A.1. In brief, no special assumptions apart from those already made are required. Nonetheless, we note that for the more general way of obtaining our key insights (in section 4.2), we do not use the functional form in eq. (8), as a result of which details regarding its parameterization do not affect our results.

[5] We recall a few basic properties of the log normal distribution: if $Z \sim \log \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}(Z) = e^{\mu + \frac{\sigma^2}{2}}$ and $\mathbb{V}(Z) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$. These expressions for the mean and variance, together with the moments required from eq. (8), will determine the scale, $\sigma^2(X_t) > 0$, and location, $\mu(X_t)$, of the log normal distribution. In fact, in our case, $\mu(X) = -\dfrac{\sigma^2(X)}{2}$.

Per-period expected utility is now

$$\mathbb{E}\Big(U\left[C\left(X_t\right)\right]\Big) = \mathbb{E}\Big[\log\left(eg_tX_t^{\alpha}\right)\Big] = \mathbb{E}\Big[\log\left(e\right) + \log\left(g_t\right) + \alpha\log\left(X_t\right)\Big], \tag{9}$$

while the objective function is

$$\max_{e\in(0,1)}\lim_{t\to\infty}\mathbb{E}\Big[\log\left(eg_tX_t^{\alpha}\right)\Big]. \tag{10}$$

The steady state of stochastic dynamical system represented by eq. (2b) is itself a probability distribution over stock levels (inducing a distribution over consumption and utility). It will turn out that the use of these functional forms allows us to make progress, in particular by allowing us to express utility explicitly in terms of stocks . In more detail, to find the form of the steady state distribution, we take the logarithms of both sides (of eq. (2b)) to obtain

$$\log(X_{t+1}) = \log(1-e) + \log(g_t) + \alpha\log(X_t).$$

Taking expectations[6] we obtain,

$$\mathbb{E}\big(\log(X_{t+1})\big) = \log(1-e) + \mathbb{E}\big(\log(g_t)\big) + \alpha\mathbb{E}\big(\log(X_t)\big). \tag{11}$$

An expression for $\mathbb{E}\big(\log(g_t)\big)$ may be derived as follows,

$$\begin{aligned}\mathbb{E}\big(\log(g_t)\big) &= \mathbb{E}_{X_t}\Big[\mathbb{E}\Big(\log\left(g_t|X_t\right)\Big)\Big]\\ &= \mathbb{E}_{X_t}\left(-\frac{1}{2}\log(v) + \lambda\log(X_t)\right)\\ &= -\frac{1}{2}\log(v) + \lambda\mathbb{E}\log(X_t).\end{aligned} \tag{12}$$

[6]Expectations in the text that are not marked by an * are to be understood to be w.r.t. the time s measure, Ψ_s (generated by the Markov chain that the stock evolution in eq. (2b) represents); while those marked * represent expectation w.r.t. the invariant measure, $\Psi^\star$, of this Markov chain (see section 4.2).

Combining these two expressions, we find that

$$\begin{aligned}\mathbb{E}\left(\log(X_{t+1})\right) &= \log(1-e) + \mathbb{E}(\log(g_t)) + \alpha\mathbb{E}(\log(X_t)) \\ &= \log(1-e) - \frac{1}{2}\log(v) + (\lambda+\alpha)\,\mathbb{E}(\log(X_t)). \end{aligned} \tag{13}$$

Denote by $\mathbb{E}^{\star}(X)$ the "steady state" expectation (in lieu of the more cumbersome $\mathbb{E}_{\Psi^*}\big(X\big)$), and similarly for $\mathbb{E}^{\star}\big(U(X)\big) := \mathbb{E}^{\star}\big(\log(X)\big)$. If $\mathbb{E}(\log(X_t))$ converges to the steady state expected stock level $\mathbb{E}^{\star}(\log(X))$, then our analysis below, involving the expected steady state utility, is rigorous. These aspects will established in the subsequent section for a more general setting. Consequently, for now, we assume the needed convergence and proceed in an intuitive way.

Taking the limit of both sides of eq. (13), and then taking the expectation of the utility function, we find that in the steady state,

$$\mathbb{E}^{\star}(\log(X)) = \frac{\log(1-e) - \frac{1}{2}\log(v)}{1-(\alpha+\lambda)}, \tag{14a}$$

$$\mathbb{E}^{\star}(U) = \mathbb{E}^{\star}(\log(C)) = \log(e) + \frac{\alpha+\lambda}{1-(\alpha+\lambda)}\log(1-e) - \left\{\frac{\log(v)}{2}\left[\frac{1}{1-(\alpha+\lambda)}\right]\right\}. \tag{14b}$$

In this set-up, the *Ecological golden rule* level of effort is the value of e that maximizes eq. (14b), the expression for long-run utility. This level of effort must balance, as in the deterministic case, the benefits of higher consumption (for a given the stock level) with the steady state (expected) level of the stock; in addition, it also must consider the steady state variance in the level of the stock. The latter is important because it is directly related to the variance of (steady state) consumption, which, because of the concavity of the utility function, reduces the expected utility of consumption.

An examination of eq. (14b) leads to several observations.[7] In the simplest case, there is no variance in the growth rate ($v = 1$, $\lambda = 0$) and we naturally recover the deterministic result. If, however, there is uncertainty in growth rate, g_t, which is in-

[7] Clearly, the results in eq. (14b) only make sense for $\alpha + \lambda < 1$. Overall, then, the results here pertain to a system in which growth of stock is only moderate (low α), and so too is the effect of stock upon variance (i.e. λ is 'small'.)

dependent of the level of the stock (i.e. $\lambda = 0$ but $v > 1$), then eq. (14b) differs from its deterministic version in eq. (6) by an additive, negative constant independent of e. This latter feature has an important and intuitive implication: for any given level of effort, utility with stochastic growth is lower than in the deterministic case (recall that $v \geq 1$), for both stock-dependent and i.i.d. growth factors.[8] This is just an expression of the risk aversion embodied in the utility function. More importantly, since this utility reduction is independent of the level of effort, it also means that the optimal level of effort is unaffected by the random nature of the growth factors.

In the more interesting case of stock-dependent variance (i.e. $\lambda > 0$ and $v > 1$), a comparison of eq. (14b) and eq. (6) yields an additional (to the downward shift discussed above) effect. The implications of this for optimal effort may be discerned by a simple comparison with eq. (6), which indicates that λ enters as an increment to the natural growth rate, α. Consequently, we find that the optimal level of effort is shifted downward by λ,

$$e^{\star} = 1 - \alpha - \lambda. \tag{15}$$

In other words, the golden rule level of effort is lower when there is stochastic growth with stock-dependent variance, unless $\lambda = 0$. When $\lambda = 0$, the variability in growth rates is independent of the level of the stock, so it makes sense for the choice of the optimal effort to be unaffected. In all other cases, though, the higher variability in growth for lower levels of the stock makes it desirable to maintain larger levels of stock. Our main result can be summarized as:

Proposition 1. *In the model of resource extraction represented in eq.* (2) *and eq.* (3)*, with growth factors log normally distributed as in eq.* (8)*, the optimal ('golden rule') level of effort is reduced as a result of uncertainty in stock if, and only if, the variance in growth rates is larger at lower stock levels.*

[8] Comparing the expression for $U^{\star}(C)$ from eq. (6b) and $\mathbb{E}^{\star}(U(C))$ from eq. (14b), we see that $U^{\star}(C) - \mathbb{E}^{\star}(U(C)) = -\frac{\lambda \ln(1-e)}{(1-\alpha)(1-\alpha-\lambda)} + \frac{\ln(v)}{2}\left[\frac{1}{(1-\alpha-\lambda)}\right] \geq 0.$

4.2 More General Probabilistic Framework

We turn now to a more general probabilistic framework, taking account of the crucial property of the growth distribution, that adverse growth shocks are more likely at lower stock levels. In terms of the distribution of growth factors, this implies that growth factors are more likely to be lower when stock levels are lower. Formally, for the system dynamics in eq. (2b), if $F_X(z)$ is the distribution function of g_t when $X_t = X$ and $\mathcal{Z} := (0, \infty)$ is the support of g_t, then $F_X(z) \leq F_{X'}(z)$, $\forall z \in \mathcal{Z}$, whenever $X \geq X^{'}$. Clearly, then, mean growth factors are also larger when stock levels are larger i.e. $\mathbb{E}_X(g) \geq \mathbb{E}_{X'}(g)$.

We indicate, following Stachurski (2003, p. 144), a general way in which one may generate a random variable (r.v.) from a distribution function which satisfies this condition. Consider a random variable independent of the stock, with ϕ its density function, and an increasing and strictly positive function of X, $\nu(X)$, and set

$$F_X(z) = \int_0^{\frac{z}{\nu(X)}} \phi(s)ds. \tag{16}$$

It follows by construction (note that F_X–viewed as a function of X– is decreasing in X) that this formulation satisfies the required condition on the distribution of g_t. To summarize, we begin with an independent (of X) r.v. with density ϕ, and, using the transformation in eq. (16), generate the required stock-dependent random variable, g_t (with distribution function F_X).

This method of construction not only yields a r.v. satisfying the probabilistic conditions required but also aids in establishing convergence, as detailed subsequently. In any case, using this method, we prove that proposition 1 holds whenever g_t has either the log normal or the weibull distribution. We provide next details for the case of the log normal distribution, and refer the reader to appendix B for the Weibull case. Let ψ represent the density function of a log normal random variable, with parameters (μ, σ^2). Consider the integral below, which is the distribution function of g_t when $X_t = X$ (i.e.

the conditional distribution of g_t):

$$F_X(z) = \int_0^{z/\nu(X)} \phi(s)ds = \Phi\left[\frac{\log\left(\dfrac{z}{\nu(X)}\right) - \mu}{\sigma}\right] = \Phi\left[\frac{\log(z) - \tilde{\mu}(X)}{\sigma}\right]. \tag{17}$$

Φ is the distribution function of a standard normal variate and $\tilde{\mu}(X) = \mu + \log(\nu(X))$, with $\nu(X) > 0$ (we work throughout with $\nu(X) = X^a$, $a \in (0,1)$, for simplicity). Thus, we have that $g_t \sim \mathcal{LN}\left(\tilde{\mu}(X), \sigma^2\right)$. Recalling the fact that if $g_t \sim \mathcal{LN}(\tilde{\mu}(X), \sigma^2)$, $\log(g_t) \sim N(\tilde{\mu}(X), \sigma^2)$, we have that $\mathbb{E}\left(\log g_t\right) = \tilde{\mu}(X) = \log(\nu(X)) + \mu$. Noting that the preceding results condition on X, we have that

$$\mathbb{E}(\log g_t) = \mathbb{E}\Big[\mathbb{E}_X\big(\log\left(g_t|X_t\right)\big)\Big] = \mathbb{E}_X\Big(\tilde{\mu}\left(X\right)\Big) = a\mathbb{E}(\log X_t) + \mu. \tag{18}$$

We note that here we have made no special assumptions on the parameter values for μ and σ. It is easy to check that using this expression for $\mathbb{E}(\log g_t)$, eq. (14b) is unchanged (after identifying μ here with $-\frac{1}{2}\log(v)$, and a with λ), and so is our main result, proposition 1. As an added benefit, it should be evident that this way of deriving the expression for $\mathbb{E}(\log g_t)$ also obviates the care necessary to ensure that the parameterization of the stock-dependent variance specification, in eq. (8c), is sensible (see also remark A.1).

To conclude, using the procedure outlined in eq. (16), we derive explicitly an expression for $\mathbb{E}\Big(\log\left(g_t\right)\Big)$ for the more general log-normal case, and find it identical to that used in deriving our green golden rule (eq. (14b)). If the convergence assumed in deriving eq. (14b) can now be shown to be valid for this more general case, then the validity of proposition 1 is rigorously established. Establishing convergence is precisely the task we take up next, and is the content of our main technical result, whose proof is presented in appendix C.

One final step remains before we are ready to state our main result. Here, we indicate how the effect of stock X_t on g_t in eq. (2b) can be de-coupled. Based upon the construction of eq. (17), it can be deduced that $\eta_t \nu(X)$ has the same distribution as g_t, when η_t is an i.i.d random variable (we consider only the log normal and weibull cases).[9] Thus, eq. (2b) may be rewritten as

$$X_{t+1} = \underbrace{(1-e)\, X_t^{\alpha}}_{\text{f}} g_t = \underbrace{f(X_t)\nu(X_t)}_{\hat{f}} \eta_t = \eta_t \hat{f}\,(X_t)\,. \tag{19}$$

Proposition 2. *Let $\mathcal{X} \in \mathbb{R}_+ \backslash \{0\}$ (not necessarily compact) and the probability structure for growth factors be as in eq. (17)-eq. (18). Then, for $e > 0$, a strictly positive distribution for X_0 (possessing a density), and under conditions weaker than in stochastic renewable resource models (see appendix C for detailed conditions):*

a. The Markov chain in eq. (19) converges to a unique, invariant distribution, Ψ^*.

b. The first and the logarithmic moments both converge i.e. $\lim_{t\to\infty} \mathbb{E}\big(X_t\big) = \mathbb{E}^\star\big(X\big) < \infty$ and $\lim_{t\to\infty} \mathbb{E}\big(\log(X_t)\big) = \mathbb{E}^\star\big(\log(X)\big) < \infty$, where $\mathbb{E}^\star(.) := \int_{\mathcal{X}} (.)\Psi^\star(dx)$. and $\mathbb{E}\left((.)_t\right) := \int_{\mathcal{X}} (.)\Psi_t(dx)$.

Proof. See Appendix C. □

4.3 Degree of Stock-dependence and Stationary stock levels

Our main finding, in proposition 1, relates effort levels to the degree of stock dependence. It may be surmised, given the monotonic relationship between effort levels and stock dependence (i.e. that $e^\star = 1 - \alpha - a$), that a similar relationship may exist between the asymptotic stock distribution and stock-dependence. More specifically, the question of interest is this: if the decision maker applies optimal effort every period, then is it the case that increasing stock dependence leads to larger (asymptotic) stocks? We answer this question in the affirmative. While intuitive, this is not a trivial result, and indeed, does not directly follow from the equation that encapsulates our main result, eq. (14b) (since it is couched in terms of $\log(X)$). We therefore work directly with the stock evolution in eq. (19).

[9] Let η_t be an i.i.d r.v. with density ϕ, as defined above. To see that $\eta_t\nu(X)$ has the same distribution as g_t, consider: $\mathbb{P}\left(\eta_t\nu\left(X\right) \leq y\right) = \mathbb{P}\left(\eta \leq \frac{y}{\nu\left(X\right)}\right) = \int_0^{z/\nu(X)} \phi(s)ds$. But the latter expression is precisely the definition of F_X, the distribution function of g_t, see e.g. eq. (17).

We begin first with some intuition, considering the function $\nu := X^a$, and assume effort is exerted optimally, per proposition 1. It is evident, from an inspection of eq. (19), that in this framework X_{t+1} is *increasing* in the degree of stock dependence, for a given η, for each t. This is clearly intuitive and follows from the assumption that growth factors exacerbate the effects of low stocks. Since these both are true $\forall t$, one anticipates the same of the invariant stock distribution. In essence, stock-dependence of the form considered here acts as an increment to the growth function, leading to altering the consumption-saving trade-off. Our task here is to formalize the notion of "X_{t+1} increasing in stock dependence $\forall t$", and to show that it also holds for the stationary distribution.

More formally, then, denoting by F_{t+1} the time $t+1$ (induced) distribution of X_{t+1}, it is straightforward to show (see appendix D) that F_{t+1} is *increasing* in a, when effort levels are optimal ($e = e^\star := 1 - \alpha - a$), in the sense of the stochastic order i.e. for $a_1 > a_2$, $F_{t+1}(.; a_2) \geq F_{t+1}(.; a_1)$. This implies that the mean stock is increasing in a as well i.e. it is the case that $\mathbb{E}\Big(X_{t+1}|X_t = x, a = a_1\Big) \geq \mathbb{E}\Big(X_{t+1}|X_t = x, a = a_2\Big)$.[10] Since these facts are true for every t, it seems sensible to anticipate that it is also the case asymptotically. It will turn out that, under almost no additional conditions on the transition, such is indeed the case. The proof is presented in appendix D, and its economic implications and connections are explored subsequently. We note, however, that it is rare in resource economics to be able to characterize these aspects of the invariant distribution. We formally state this result as

Corollary. *(To Proposition 1) In the model of resource extraction constituted by eq.* (19)*, eq.* (2) *and eq.* (3)*, for the optimal effort level, the invariant distribution is stochastically increasing in a i.e.* $a_1 > a_2$ *implies* $\Psi^\star_{a_1}$ *(first order) stochastically dominates* $\Psi^\star_{a_2}$*. Thus, expected (optimal) stationary stock is larger whenever growth factors exacerbate the effects of low stock levels. In particular, then,* $\mathbb{E}^\star\Big(X|a > 0\Big) > \mathbb{E}^\star\Big(X|a = 0\Big)$.

[10] A family of distribution functions $\{F_t(a)\}_{a\in(0,1)}$ is stochastically increasing (SI) in the parameter a iff for $a_1 > a_2$, $F_{a_1} \leq F_{a_2}$. Since the latter is the definition of first order stochastic dominance, F_{a_1} (first order) stochastically dominates F_{a_2}, denoted as $F_{a_1} \succeq F_{a_2}$. Consequently, $\{F_t(a)\}_{a\in(0,1)}$ stochastically increasing in a implies $\int XdF(a_1) \geq \int XdF(a_2)$, or alternatively, $\mathbb{E}(X_{t+1}(a_1)) > \mathbb{E}(X_{t+1}(a_2))$, where $X_{t+1}(a)$ is the Markov chain in eq. (19), viewed as a function of the parameter a.

4.4 Discussion

We note that our problem bears a resemblance, albeit faint, to the deterministic problem in Chichilnisky et al. (1995), whose objective is to find feasible optimal paths for long run utility (when utility is a function of natural stock and man-made capital). Theirs is a problem closely related to that of economic growth, leading to their calling the optimal consumption path for their problem the 'Green golden rule'. Our problem is different in fundamental ways, not least of which is the constant-over-time effort levels. Nonetheless, our optimal effort levels are always feasible and sustainable and our optimality criterion bears a resemblance to theirs. Given these similarities, and the ecologically-based motivation for our stock dynamics, we term our optimal effort levels the 'Ecological golden rule' level of effort.

Our main result, proposition 1, turns out to have an intuitive and interesting economic interpretation. Recall that effort levels–being time invariant– parameterize the (time t and long-run) stock distribution, and that, it turns out, in a particularly simple way: every level of effort may be viewed as leading to a lottery over (long-run) consumption. In this interpretation, the main question is one related to choice of the optimal lottery. Indeed, it follows that there is a specific form of the mean-variance trade-off (over consumption) involved in this choice: increased effort directly increases the mean but also, equally directly, the variance. This direct increase in variance is an additional source of disutility to the risk averse decision maker.

The generalization in section 4.2 provides further insight into the precise nature of the altered trade-offs: stock dependent growth factors that are lower with lower stocks alter the nature of the calculus between the present and the future, in essence rendering future stocks more valuable. In fact, for the power function, it serves as an increase in the growth rate. Thus, in comparison to the case of stock-independent growth factors, there is a very transparent additional incentive to reduce extraction.

Our finding regarding increased stationary stock size, which follows from the altered calculus discussed above, is interesting, and has connections to the literature on 'precautionary wealth accumulation' in dynamic stochastic settings. This result may be interpreted as stating that when faced with increased "risk", represented by increases in the stock-dependence parameter a, the decision maker responds by accumulating larger stocks. The intuition underlying our finding is similar to that driving the finding of "precautionary wealth accumulation" by individuals who face increased risk at every stage of the life-cycle (see e.g. Huggett (2004) and references therein): in both cases the 'savings function' is increasing in risk (in our case, $s := 1 - e^\star$ is increasing in a).[11]

5 Conclusions

This paper is concerned with analyzing the implications of a relationship long held as important by ecologists: that stock-dependent uncertainty in growth rates are important for understanding population dynamics, and therefore for managing renewable natural resources with these dynamics. A simple but rigorous stochastic dynamic bio-economic model is employed, with consumption of the renewable resource providing utility while depleting the stock of the resource, which is replenished with a random growth factor which varies with stock level. In contrast to many existing studies that are focused on maximizing discounted utility, we are concerned with maximizing long-run utility, emphasizing the connection with sustainability.

Our main finding is that extraction effort is *lower* in cases where the variance of natural growth rate of stock is higher for lower stock levels, an important feature of many ecological systems. This is, to our knowledge, the first formal analysis of economic management of ecological systems with such a feature. An important contribution of this paper lies in identifying a new mechanism, one ecologically based, governing harvest effort in dynamic renewable resource systems. Previous mechanisms have been focused on substitution, between man-made and natural capital (in a deterministic framework, in Chichilnisky et al. (1995)) and between bio-diversity and consumption (in a stochastic framework, Li et al. (2001)). What we identify is an entirely new mechanism: an intrinsic

[11] While the modeling frameworks in Huggett (2004) and the cited literature are different (maximizing expected discounted consumption versus expected asymptotic consumption), and so too are the notions of stochastic order used (increasing and convex instead of the simpler first order stochastic dominance we use), fundamental aspects of the problem are similar.

link between stock level and its variability. While our results are derived in the context of a specific setting, they are plausible, and indeed intuitive, enough that it is anticipated that they hold in broad generality.

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Appendix A Basic Setup

We gather here a few miscellaneous details regarding certain aspects of the model discussed in section 4.1.

Remark A.1. *(Set-up for the Specific Model) We assume that* $X_t \in \mathcal{X} \subset \mathbb{R}_+ \backslash \{0\}$, $\mathcal{X}$ *compact. Weaker assumptions are made for the more general probabilistic formulation in section 4.2 (see appendix C). This aspect, together with the conditions outlined in footnote 7, ensure that the condition for positive variance (from eq. (8c)),* $X \leq v^{\frac{1}{2\lambda}}$*, is easily met, at least for* $\nu \geq \left(\overline{X}\right)^{2\lambda}$*.*[12]

In addition, it may seem that the requirement that the variance in growth rates diverges as the stock approaches zero is very strong, but note that for small values of X_t, $\mathbb{V}(X_t + 1|X_t) \approx X_t^{\alpha - 2\lambda}$*, which can vanish if* $\alpha > 2\lambda$*, so the stock is well-behaved in the neighbourhood of* 0*.*

Appendix B Other Distributions

We indicate an extension of the result for the log normal case in section 4.2 to the Weibull distribution. Let now ϕ be the density function of a single-parameter weibull distribution,[13] which implies that $\phi(s; \beta) = \beta s^{\beta-1} \exp\left\{-(x)^{\beta}\right\}$ where $\beta > 0$. Following an identical procedure as for the log normal case (i.e. using this weibull distribution and the function ν), we obtain the needed stock-dependent r.v. g_t as

$$F_X(z) = \int_0^{z/\nu(X)} \phi(s)ds = 1 - \exp\left(\frac{-z}{\nu(X)}\right)^{\beta}.$$

More explicitly, this expression implies that $g \sim \text{Weibull}\,(\beta, \nu(X))$. For $T \sim \text{Weibull}(\beta, \nu(X))$, it is the case (see e.g. White (1969, p.375)) that $\log T \sim$ log-Weibull, with the distribution function $F_X(t) = 1 - \exp\left(-\exp\left[\beta(t - \log \nu(X))\right]\right)$. More importantly for the

[12] In other words, provided $X \in (0, 1]$, then the variance of g in eq. (8c) is positive $\forall\ \lambda < 1$ and $v > 1$. In fact, the key assumption is that X is uniformly bounded away from 0 (i.e. $X_t > M > 0$, $\forall t$), a standard assumption in many dynamic stochastic economic models of renewable resources.

[13] The standard two-parameter weibull distribution has density $\phi(s; \beta, \theta) = \frac{\beta}{\theta}\left[\frac{s}{\theta}\right]^{\beta-1} \exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}$ where $\theta, \beta > 0$ where for simplicity, and w.l.o.g, we set $\theta = 1$.

present purpose, its moments maybe computed through a simple transformation[14] (see White (1969, pp 375-76) and also Johnson et al. (1994, Vol. 1, pp 635 and Vol 2. pp 2-3)) yielding

$$\mathbb{E}(\log g) = \frac{-\gamma}{\beta} + \log(\nu(X)) = \frac{-\gamma}{\beta} + a\mathbb{E}(\log X). \tag{B.1}$$

Thus, $\mathbb{E}\left(\log g_t\right)$ is again of the same form as in eq. (12), and the remarks following it are directly applicable.

Remark B.1. *We note that unlike for the special functional form for g_t in eq. (8), no undue restrictions on the stock are required for positive variance. To see this, note that the expression for the variance of g_t is*

$$V\left(g_t\right) = \begin{cases} \left[e^{\sigma^2} - 1\right]\left[e^{2\tilde{\mu}(X)+\sigma^2}\right], & \text{if } LN \\ \nu\left(X\right)^2 \left[\Gamma\left(1+\frac{2}{\beta}\right) - \left\{\Gamma\left(1+\frac{1}{\beta}\right)\right\}^2\right], & \text{if Weibull} \end{cases}. \tag{B.2}$$

Clearly, no restrictions are required for the case of the Weibull distribution, while for the log normal case, the only restriction needed, $X > \exp\left(-\frac{(\sigma^2+\mu)}{a}\right) > 0$, is consistent with our assumption of a strictly positive stock.

Appendix C Proof of proposition 2

In a dynamic stochastic setting, it is of substantial importance to understand when the sequence of distribution functions, $\{\psi_t\}$, associated with the evolving random variables, $\{X_t\}$, converge to a unique distribution, ψ, called the invariant distribution. We note that when the shocks are stock-dependent, there are few general approaches or results which are directly applicable. The overall strategy of our proof is split into two steps: converting the SRS with stock-dependent growth factors into one with i.i.d growth factors and proving uniqueness of the invariant distribution (following the approach of Stachurski (2009, §11.3.5)) and convergence of moments (following Tweedie (1983)).

[14] γ is the Euler-Mascheroni constant, obtained by differentiating the Γ function i.e. $\Gamma^{'}(1) = -\gamma$.

Consequently, we indicate how the key conditions required in the relevant theorems are satisfied by our model (details of verification of key Assumptions below are available upon request).

We prove rigorously here that the Markov chain $X_{t+1} = (1-e)g_t X_t^{\alpha}$, with $g_t \geq 0$ a state-dependent random noise, $X_t \in \mathcal{X} \subset \mathbb{R}_+^{\setminus\{0\}}$, $e \in (0,1]$, $1 > \alpha > 0$, has a unique invariant distribution. Let $f^e(x) = (1-e)X_t^{\alpha}$, implying that $X_{t+1} = g_t f^e(X_t)$, where as in section 4.2 and Appendix B, g_t is such that $F(g; X) \leq F(g; X')$ whenever $X \leq X'$.[15] This can be recast into a model with i.i.d errors, as already discussed

$$X_{t+1} = \underbrace{f^e(X_t)\nu(X_t)}_{\hat{f}}\,\eta_t = \hat{f}^e(X_t)\,\eta_t := F_e(x,\eta)\,. \tag{C.1}$$

We state next the key assumptions under which the uniqueness of the invariant distribution holds, before indicating how these conditions hold for our setting. We note that unless otherwise explicitly stated, the state space $\mathcal{X} \subset \mathbb{R}^+ \setminus \{0\}$ is not compact, $e \in (0,1)$ (the boundary cases are excluded to focus on cases of most interest), and we work with the function $\nu(x) = x^a$ with $0 < a < 1$ s.t. $\alpha + a < 1$. Thus, $X_{t+1} = (1-e)\,X_t^{\alpha+a}\eta_t$, with $X_t + 1$ concave in X_t (for given e and η). While not essential, this functional form will substantially simplify our proofs.

We introduce the required objects related to the Markov chain along with some standard notation (see e.g. Hopenhayn and Prescott (1992); Stachurski (2009)). Denote by $\mathcal{P}(\mathcal{X}), \mathcal{B}(\mathcal{X}), b(\mathcal{X})$ the set of all probability (measures) distributions, Borel sets, and bounded functions respectively on the state space $\mathcal{X}$, whose arbitrary members are denoted μ, B and h. Associated with a Markov chain is a stochastic kernel, $P : \mathcal{X} \times \mathcal{B}(\mathcal{X}) \to [0,1]$, and a Markov operator, $\mathbf{M} : b(\mathcal{X}) \to b(\mathcal{X})$: the former, $P(x,B)$, is interpreted as the probability that $X_{t+1} \in B$ with $X_t = x$; the latter, $(\mathbf{M}h)(x) := \int h(y)P(x,dy)$, is interpreted as the expectation of $h(X_t)$ conditional on $X_0 = x$.[16]

[15] The stochastic recursive system (S.R.S) in eq. (C.1) corresponds to the case of "growth with state-dependent" shocks in Stachurski (2003, §4.2), to which we direct the readers for additional details.

[16] In fact, this is the so-called "right" Markov operator, with the "left" Markov operator, $\mathbf{M} : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$, defined as $(\mu\mathbf{M})(B) := \int P(x,B)\,\mu(dx)$, and interpreted as "shifting" the distribution of X one-period forward. The left Markov operator is used in appendix D.

The following assumptions hold for each $e \in (0,1)$:

Assumption 1. *Shocks η_t are i.i.d, with density $\phi > 0$, with $\mathbb{E}(\eta)$ and $\mathbb{E}\left(\frac{1}{\eta}\right) < \infty$.*

Assumption 2. *The stochastic kernel, P, generated by the Markov chain in eq. (14b), admits a density i.e. $P(x, dy) = p(x, y)\, dy$ for some "density kernel", p.*

Assumption 3. *There exists a positive function g defined on $\mathcal{X}$ such that $\int g(y)dy > 0$ and $p(x, y) \geq g(y) \forall x \in C$ and $y \in \mathcal{X}$, with C any compact subset of $\mathcal{X}$. In addition, $\int_C g(y)dy > 0$.*

Assumption 4. *$\exists$ a positive function V on $\mathcal{X}$ and constants $0 < b_1 < 1$, $0 < b_2 < \infty$ such that $\mathbf{M}V(x) \leq b_1 V(x) + b_2$, $\forall x \in \mathcal{X}$.*

Assumption 5. *The Markov process in eq. (19) is $\psi-$irreducible, for some measure ψ.*

Remark C.1. *We note that under Assumption 2 and 3, the Markov chain in eq. (C.1) is aperiodic while under Assumptions 2-4, it also satisfies the conditions pertaining to "drift to a small set" (Stachurski (2009, Lemma 11.3.31)). Further, note that the parameter e does not directly affect these (or any other) conclusions, primarily since it enters X_{t+1} independently of both X_t and η_t (and in a continuous fashion i.e. F_e is continuous in e).*

We turn next to verifying the assumptions in our case, before stating our main result. Taking Assumption 1, finiteness of mean is evidently true for both the log normal and the weibull case and clearly both distributions have densities which are strictly positive on $\mathbb{R}_+$. Next, the fact that $F_e > 0$ for $X > 0$ (for a given η and $e > 0$) and Assumption 1 together imply that, for every t, X_{t+1} can be represented by a so-called "density kernel" $p(x, y)$, verifying Assumption 2 (see Stachurski (2009, Thm 8.1.3)).[17] Next, it is known that if Assumption 3 holds (along with Assumptions 1 and 2), then the Markov chain $\{X_t\}$ is aperiodic. It can be shown that the function $g := \delta\mathbb{I}\{C\}$, with $\mathbb{I}\{C\}$ an indicator function for the set C, $\delta := \min\{p(x, y) : (x, y) \in C \times C\}$, and p the density kernel from

[17] A density kernel for a Markov chain represents the density function for a Stochastic kernel, P, . For Markov chains, the density kernel $p : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ is represented as $p(x, y)$ and interpreted as a density function for each fixed $x \in \mathcal{X}$ (see Stachurski (2009, Def 8.1.1)).

Assumption 2, satisfies all the required properties. Next, for $V(x) := |x|$, it can be seen that Assumption 4 holds, with $b_1 = \alpha + a < 1$ and $b_2 = |\log(1-e)| + \mathbb{E}(\eta) < \infty$. Finally, the irreducibility condition is defined as $\mathbb{P}(X_t \in B, t \geq 1 | X_0 = x)$, and it can be easily verified by considering X_1 i.e. $\mathbb{P}(X_1 \in B | X_0 = x) \equiv \mathbb{P}\left(\eta \in \frac{B}{(1-e)\, x^{\alpha+a}}\right) > 0$ for the Lebesgue measure (from Assumption 1).

The key result of this section is

Proposition C.1. *Under Assumptions 1-5, the Markov chain in eq.* (C.1) *has a unique stationary distribution* ψ, *for each* $e \in (0,1]$. *In addition, it is the case that* $\int_{\mathcal{X}} \log(x)\psi(dx) =: \mathbb{E}^{\star}(\log X) < \infty$ *and* $\int_{\mathcal{X}} x\psi(dx) =: \mathbb{E}^{\star}(X) < \infty$

Proof. This is Theorem 11.3.36 in Stachurski (2009). □

Remark C.2. *We note that the finiteness of moments follows from two key facts,* $\mathbb{E}^{\star}(V) < \infty$ *and* $V \geq |\log X|$.[18]

Remark C.3. *(Harris Chain) It is a standard result that an irreducible markov chain that admits an invariant distribution is positive Harris recurrent (see e.g. Bhattacharya and Majumdar (2007, Thm C9.3(ii))). Consequently, the Markov chain generated by the S.R.S in eq.* (19) *is positive Harris recurrent, implying the obvious (in view of proposition C.1) fact,* $\psi(\mathcal{X}) < \infty$.

Our final result for this section is

Proposition C.2. *For the Markov chain in eq.* (C.1), *under the conditions in proposition C.1 and the results of remark C.3, it is the case that* $\lim_{t\to\infty} \mathbb{E}(.) := \lim_{t\to\infty} \int_{\mathcal{X}} (.)\psi_t(dx) = \int_{\mathcal{X}} (.)\psi(dx) =: \mathbb{E}^{\star}(.)$, *for* $(.) = X$ *and* $(.) = \log X$ *(for each* $e \in (0,1)$*).*

Proof. This is Theorem 2 in Tweedie (1983). □

Appendix D Ordering of Invariant Distributions

Understanding when time-t, and invariant, distributions are ordered in terms of parameters, whenever the transitions are, is an important question in dynamic stochastic

[18] For the last inequality, if necessary, the state space may be redefined w.l.o.g. to be $(1, \infty)$, indicating that $\log X > 0$ (as will be assumed for proposition C.2).

economic models (see e.g. Huggett (2003) for several examples). We are able to show, under very minimal conditions on the transition, that such ordering holds on the invariant distribution in our case i.e. we show that larger values for a lead to "larger" invariant distribution (implying also naturally larger mean stocks). The only additional condition needed is

Assumption 6. *The state space $\mathcal{X} \subset \mathbb{R}_+ \setminus \{0\}$ is compact.*

Remark D.1. *Note that since F_e is increasing (and continuous) in X, the stochastic kernel associated with eq. (C.1) is said to be* increasing*. Thus, the "left" Markov operator (see footnote 16) is also increasing. From the discussion in section 4.3, it is evident that the time t stochastic kernel, P^t, is stochastically increasing in a (i.e. $P^t_{a_1} \succeq P^t_{a_2}$), implying that $\mathbf{M}^t_{a_1} \succeq \mathbf{M}^t_{a_2}$ for $a_1 > a_2$.*[19]

Proposition C.1 and remark D.1 together yield our main result for this section

Proposition D.1. *Under proposition C.1, and whenever $\mathbf{M}^t_{a_1} \succeq \mathbf{M}^t_{a_2}$ for $a_1 > a_2$, then it is the case that $\psi_{a_1} \succeq \psi_{a_2}$, where $\lim_{t\to\infty} \mathbf{M}^t_{a_1} = \psi_{a_1}$ (similarly for a_2). Clearly, this implies that $\mathbb{E}_{\psi_{a_1}}(X) \geq \mathbb{E}_{\psi_{a_2}}(X)$.*

Proof. This is Corollary 3 in Hopenhayn and Prescott (1992). □

Remark D.2. *To see that $F_{t+1}(a)$, the distribution induced by the stochastic kernel P, is stochastically increasing (SI), consider two values for a, a_1 and $a_1 + \epsilon$. If F_{t+1} is SI in a then it must be that $1 - F_{t+1}(.; a_1) \leq 1 - F_{t+1}(.; a_1 + \epsilon)$. That this is the case follows from:*

$$\mathbb{P}\Big(X_{t+1} \geq b | X_t = \gamma, a = a_1 + \epsilon\Big) \equiv \mathbb{P}\left(\eta_t \geq \frac{\frac{b}{\gamma^\alpha}}{(\alpha + a_1 + \epsilon)\,\gamma^{a_1+\epsilon}}\right)$$
$$\geq \mathbb{P}\left(\eta_t \geq \frac{\frac{b}{\gamma^\alpha}}{(\alpha + a_1)\,\gamma^{a_1}}\right) := \mathbb{P}\Big(X_{t+1} \geq b | X_t = \gamma, a = a_1\Big).$$

Remark D.3. *We note that we have assumed compact state space only for proposition D.1. A proof of ordering of the invariant distribution without using compactness*

[19] Recalling the definition of $\mathbf{M}$ from footnote 16, $\mathbf{M}^t_{a_1} \succeq \mathbf{M}^t_{a_2}$ for $a_1 > a_2$ is to be read as $\left(\mu\mathbf{M}^t_{a_1}\right)(B) := \int P^t_{a_1}(x, B)\,\mu(dx) \succeq \left(\mu\mathbf{M}^t_{a_2}\right)(B) := \int P^t_{a_2}(x, B)\,\mu(dx), \forall \mu \in \mathcal{P}(\mathcal{X})$.

can can presumably be provided by extending the analysis in Huggett (2003), using the fact that our S.R.S in eq. (C.1) *also satisfies all the conditions needed for ensuring continuity in a,*[20] *an aspect we have not in fact used in proposition D.1. However, for brevity and simplicity, we eschew this generalization.*

[20] A proof of this fact is available upon request. Note that this simply involves checking that our S.R.S satisfies all the conditions for Lemma 2 and Proposition 2 in Van and Stachurski (2007).