

Grazing Games*

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Abstract

Grasslands used for domestic livestock are often the common property of several owners and are typically characterized by complex ecosystem dynamics. We account for both these aspects by modeling the problem as a differential game where each farmer maximizes profits, given the dynamics of livestock and grass interaction. We compare the first best outcome obtained when farmers cooperate with that for non-cooperative farmers. By accounting for grassland dynamics, we challenge conventional results from the common pool literature. Although we do identify situations where the standard result holds — a tragedy of the commons unless farmers cooperate — we also find examples where a tragedy is exacerbated due to ecosystem dynamics as well as cases where a non-cooperative regime under-exploits the grassland compared to the first best outcome.

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1 Introduction

Grasslands cover about 40 percent of the earth's surface and are found in every region of the world — excluding Greenland and Antarctica. Many societies depend on grasslands for their livelihood and use them for domestic livestock. It is therefore problematic, especially in the developing hemisphere where there are little opportunities for resource substitution, that no less than 50 percent of the grasslands are degraded today. (White, Murray and Rohweden 2000.)

Such resource degradation is traditionally imputed to ill-defined property rights. Grasslands are often the common property of several farmers. The group can exclude any outsider, but amongst themselves, they are rivals and unless the users manage to cooperate in order to sustain a first best solution, a *tragedy of the commons*¹, here in the form of over-grazing, is to be expected. The prevalence of common pool resources and the inefficiencies often associated with them have given rise to an extensive literature devoted to identifying factors facilitating cooperative outcomes. (For overviews see Ostrom (1990, 2000) and Baland and Platteau (1996).)

Most of the common pool resource models assume that the resource shared can be properly characterized by simple dynamics, for example by some concave growth function. However, natural resources, including grasslands, often involve complex dynamics with ecological feedbacks and multiple species interactions, which imply that they may have multiple stable states and threshold effects. (Steffen 2004.) For example, above some critical level of grazing pressure, a grassland may undergo a sudden change, a so-called flip and end up in a completely different state, a degraded state.

Previous research shows that it is very challenging, also for a social planner, to manage such resources because even marginal changes can cause radical ecosystem transformations. (See, for example, the seminal paper by Ludwig, Jones and Holling (1978) and the papers by Crépin (2002), Brock and Starrett (2003), Crépin (2003),

Mäler, Xepapadeas and de Zeeuw (2003), and Janssen, Anderies and Walker (2004).) Hence it is possible that grassland degradation also can be imputed to such threshold effects. Several ecological models of grasslands account for this ecological complexity. (See for example Janssen Anderies and Walker (2004), Perrings and Walker (1997)) However, they fail to completely capture the rivalry between resource users, and thus do not account for all economic aspects of the problem.

The purpose of this paper is to provide a more complete picture of the over-grazing problem by combining a relevant institutional and economic structure with non-linear ecosystem dynamics. In particular we want to test the robustness of the conventional result, i.e. that a non-cooperative regime over-exploits the grassland compared to the first best cooperative outcome. We model our problem as a differential game where farmers maximize profits, given the dynamics of livestock and grass interaction and given the existence of multiple users. We compare the first best outcome obtained when farmers cooperate with that for non-cooperative farmers.

By accounting for grassland dynamics, we challenge conventional results from the common pool literature. Although we do identify situations where the conventional results hold, we also identify situations where the tragedy is exacerbated due to the non-linear dynamics — a non-cooperative regime exploits the grassland enough to cause an ecosystem flip whereas a cooperative regime does not — as well as situations where a non-cooperative regime under-exploit the resource — a cooperative regime sustains less grass in steady state compared to the non-cooperative regime.

Our results are obtained for farmers using open loop strategies. In an open loop Nash equilibrium, farmers choose their strategies at the very beginning of the time span and commit to these. An open loop strategy is a proper description of reality if farmers do not receive any new information in the course of the game. In most situations this restriction may seem extreme and one may prefer strategies, which are conditioned on the current state - so called Markovian strategies. In fact, if the purpose of this paper had been to derive an analytical solution to a specific problem we would

be worried about using open loop strategies. As it is now this is not the main purpose. Instead we want to show that the over-grazing problem cannot be properly analyzed if one only looks at one of the two aspects — property rights and complex ecosystem dynamics — separately. For this purpose we argue that it is enough to look at the simplest of information structures. Moreover, it is challenging to find the solution to the Markovian specification also in simpler games. In our problem the non-linearities increase the level of complexity even more. For these reasons we have chosen to look at the open loop solution.

According to the game theoretic literature, a tragedy of the commons is to be expected when users cannot use history dependent strategies, so called *trigger strategies*, to enforce optimal outcomes. Dutta and Sundaram (1993) challenged this result when they showed that even if agents used Markovian strategies conditional on state only, they may under-exploit the resource compared to the first best outcome. However, Dutta and Sundaram's result hinges on specific initial conditions and strategies that are sub-game perfect, thereby indirectly conditioned on action history. In our model we assume strategies that are not even sub-game perfect. Still we find under-exploitation, which means that besides initial conditions, our result hinges solely on resource dynamics.

Our study is closely related to the article by Mäler, Xepapadeas and de Zeeuw (2003) on shallow lakes that can flip between a clear and a turbid state. Mäler et al. studied a case when it was optimal to manage the lake in a clear state. When there were several communities sharing the lake, situations where the lake flipped to the turbid state could occur. Given such a situation, they elaborated on how to correct this specific type of inefficiency. Our approach is different. We do not have the intention to characterize and analyze a specific situation. Instead, our study aims at providing a more complete picture of various mechanisms behind grassland degradation. Moreover, although both studies deal with non-linear dynamics, there are also differences concerning ecosystem modeling. For one, their control variable — the load

of nutrients — has a direct effect on the resource, whereas our control variable — the animal off-take — only indirectly affects grass biomass. We also consider alternative ways to model species interaction, which turns out to have drastic implications for the results.

The remainder of the paper is organized as follows. Section 2 introduces ecosystem dynamics and management models associated with grazing. In section 3, we proceed to present and analyze outcomes resulting from different management regimes. Section 4 discusses the results.

2 Ecosystem dynamics and management

There are many kinds of grasslands in the world such as the moist prairies of the temperate hemisphere, the tropical savannas with mixture of woody vegetation and grass, and the dry alpine grasslands of China. These grasslands differ with respect to species characteristics, species interactions and ecological dynamics but they also show some common features. Grasslands typically exhibit complex ecosystem dynamics with ecological feedbacks and different states of encroachment: woody vegetation or dry desert states contrast with highly productive grassy states that are more attractive for raising livestock. (Scholes, 2003.) Moreover, there are clear indications that the land cover in many grassland landscapes have changed drastically towards less grassy states in recent years. (Lepers et al, 2005.) Although there are different underlying mechanisms, heavy grazing by domestic livestock seems to be playing a major role. Scholes and Archer's (1997) and Scholes (2003) pointed out that sustained heavy grazing by domestic livestock often combined with reductions in fire frequency have resulted in abrupt increase of woody plant and grassland degradation. Zhou et al. (2005) state that overgrazing by sheep seems to lead to desertification in many grasslands of China. In the alpine grasslands of China in particular, the principal causes of grassland degradation seems to be long-term overgrazing combined with the

destruction by rodents and climate change. (Akiyama & Kawamura, 2007.)

The diversity and complexity of grassland ecosystems indicate that it is difficult to grasp all aspects of grassland dynamics in one model. Scholes and Archer's (1997) exemplified that statement for savannas. So what are the important aspects that one should focus on? There is no universal answer to that question and the key elements must vary with the model's purpose. In this exercise, besides the institutional structure we focus on grazing and by doing so there is one feature besides non-linearities that we need to pay some extra attention to: the effect of grassland quality on animal performance. There is a built-in negative feedback on stock quality and quantity if grass becomes too scarce. The intensity of this feedback varies with livestock and grassland characteristics. The presence of browsers and the availability of alternative food sources also play a role. (See e.g. Van Keuren and Parker, 1967, Bement, 1969, Hart et al, 1988 or Manley et al, 1997.) It is difficult to find one typical form of this feedback. We have dealt with this problem by analyzing two representative cases, 1) a case in which the growth rate is strongly correlated with grazing — a non-linear negative feedback from available grass to animal performance— and 2) the limit case where there is no such feedback. We believe that these two representative cases can grasp different levels of the intensity of the feedback. Moreover, we do not have the purpose to model a specific grassland ecosystem. Instead we want to show how important it is to represent the feedback to animal performance and other specifications correctly. For that purpose, studying a case with a rather strong feedback and compare it to the limit case with no feedback is quite instructive.

2.1 *Grassland dynamics*²

We consider n identical farmers, who together have exclusive access to a piece of grassland where they let their livestock³ graze. Let $s_i(t)$ represent farmer i 's private non-negative stock of animal at time t . The total stock of animal on the land at time t is $s = \sum_{i \in I} s_i$, where $I = \{1, 2, \dots, n\}$. We let livestock growth follow a logistic function

with a constant carrying capacity⁴, k and an intrinsic growth rate, b . The animal off-take (harvest), h_i , which can be positive or negative⁵, that farmer i makes on her stock also affects livestock growth. We let h represent total animal off-take, $h = \sum_{i \in I} h_i$.

There are different ways to model how livestock assimilate eaten grass as growth. (Turchin, 2003.) In the model we use, there are two obvious ways, by which the available amount of grass could affect livestock growth: through effects on carrying capacity, k , and through the intrinsic growth rate, b . For simplicity and transparency we choose to focus only on the effects through the intrinsic growth rate b .

Let us first consider a situation with no feedback to animal performance. In this case the intrinsic growth rate is constant. We let dots denote time derivatives. The equations of motion for farmer i 's stock of animal and for the total stock of animal are respectively:

$$\dot{s}_i = s_i b \left(1 - \frac{s}{k}\right) - h_i, \quad (1)$$

$$\dot{s} = sb \left(1 - \frac{s}{k}\right) - h. \quad (2)$$

Henceforth, we illustrate the negative feedback case by letting the intrinsic animal growth rate be expressed as

$$b(g) = b \frac{g^2}{x^2 + g^2} \quad (3)$$

where g denotes grass and x is the so-called half saturation grass biomass per unit of grass. This sigmoid functional form is referred to as a Holling-type III functional response (Holling 1959), which is often used to model species interactions through feeding (Murray 2002, Turchin 2003). For a low grass biomass, animals spend most of their time searching for grass, for higher grass biomasses, they spend most of their time ingesting it. Put differently, when the non-negative grass biomass g , is below some critical value, x , the growth rate increases in a convex way with grass and for grass biomasses above this critical value, the growth rate is still increasing in grass biomass, but is now concave. The equations of motion for farmer i ' animal stock and

for the total stock are then respectively:

$$\dot{s}_i = s_i b \frac{g^2}{x^2 + g^2} \left(1 - \frac{s}{k}\right) - h_i, \quad (4)$$

$$\dot{s} = s b \frac{g^2}{x^2 + g^2} \left(1 - \frac{s}{k}\right) - h. \quad (5)$$

So far grass biomass has been considered constant, but this is only part of the story because grazing also affects grass growth. We use a simplified version of the model presented in Janssen, Anderies and Walker (2004) to model grass dynamics.⁶ Let parameter a denote a measure for grass re-growth potential. Equation (6) summarizes grassland dynamics. The first part describes grass growth in grassland where livestock is absent and the second part describes the effect of grazing. We abstract from wild animals' grazing.

$$\dot{g} = (a + g)(1 - g) - s \frac{g^2}{x^2 + g^2}. \quad (6)$$

Essentially we study two systems. The grass equation (6) is the same in both systems but the equations for livestock dynamics differ — compare (1-2) with (4-5). In a system where the intrinsic growth rate depends on the amount of grass available, there is a negative feedback effect dampening grazing pressure. The grassland becomes degraded due to grazing but the degraded grassland also limits animal growth. In a system where intrinsic growth rate is constant there is no such dampening effect. We expect that these two systems differ with regard to their resilience. Resilience is here referred to as the ecosystem's capacity to tolerate disturbances without collapsing into an alternate degraded state. Above some critical grazing pressure, due to the non-linearities, the grassland may undergo a sudden change, a so-called flip — the grassland goes from being grass abundant to grass scarce and may even become a desert. For a given system, a high re-growth potential a , and a high half saturation point, x , capture a high resilience. For a given grass biomass, the higher the values of these parameters, the higher is the steady-state stock of animal that the grassland can sustain without collapsing into an alternate state with low grass biomass.

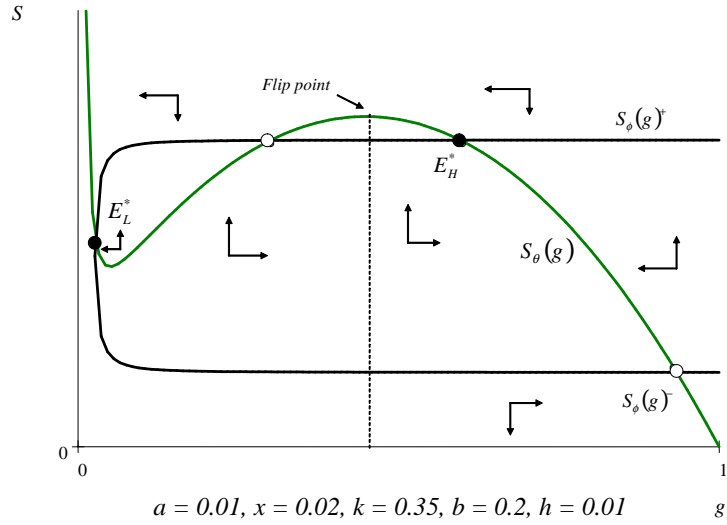


Figure 1: Phase diagram of the livestock - grass interaction for system (5,6).

Farmers determine the size of their own stock, and thereby also indirectly the effect on grass dynamics through their choice of animal off-take. However, to illustrate the grass–livestock interaction and the role of the threshold we temporarily assume a constant harvest. Figure 1 depicts the system’s dynamics (5, 6) in a phase diagram where $S_\phi(g)$, shows the nullcline for livestock with a negative feedback effect and $S_\theta(g)$ shows the nullcline for grass.

When grazing is non-linear and resilience is relatively low, $S_\theta(g)$ is convex-concave. This reflects a situation where the grassland may flip between alternate stable states.⁷ There are four steady states, situated at the intersection of the curves. The state denoted E_H^* represents a stable steady state with high grass biomass, whereas E_L^* represents a stable state with low grass biomass. The other states are unstable. For the harvest level chosen, initial grass biomass determines in which of the stable states an ecosystem will end up. To get an understanding of the flipping mechanism, consider a high grass biomass stable state and that, for some reason, the harvest level decreases. The upper stable part of curve $S_\phi(g)$ shifts upward. If the change in grazing pressure is high, the ecosystem cannot cope with this change and the grassland flips to a low

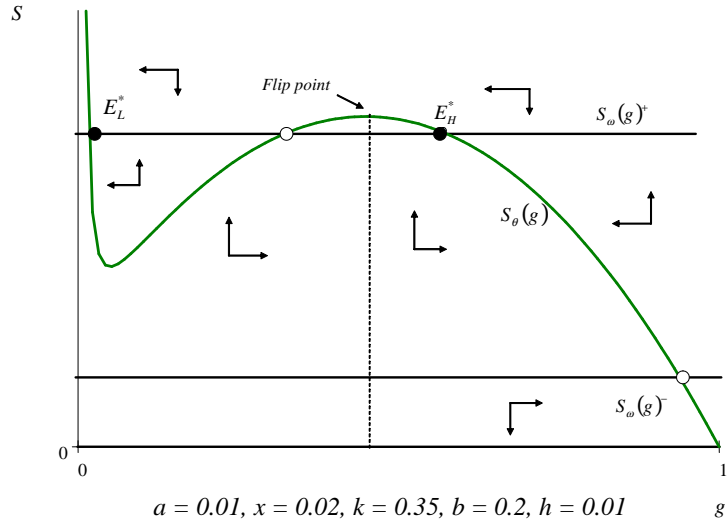


Figure 2: Phase diagram of the livestock - grass interaction for system (2,6).

grass biomass steady state. For the system (2, 6), the phase diagram, which is depicted in Figure 2, is similar except for the nullcline for livestock, which instead is represented by two parallel horizontal lines. This implies that the system (2, 6) also has two stable steady states, one grass dominated and one grass scarce. Both yield a higher stock of animal and a lower stock of grass compared to the system (5, 6).⁸

As even marginal changes can have drastic consequences for the ecosystem, it is reasonable to think that the non-linearity present in the ecosystem is part of the explanation for the degradation of grasslands observed. However, we also need to consider the economic aspect.

2.2 Economics of a common grassland

The level of animal off-take is typically not exogenously given. Instead a decision maker chooses animal off-take to maximize her objective function. We focus on the farmers' individual objectives, which we assume to be profit maximization. Farmers attach no value to the grassland besides those associated with grazing. Each farmer can sell her animal off-take at a constant price p . The cost of animal off-take, $c(h_i)$ is

convex, where $c' = \frac{dc}{dh_i} > 0$ and $c'' = \frac{d^2c}{dh_i^2} > 0$. There is also a cost for holding animals⁹, which is linear in stock size, with γ being the cost per animal.¹⁰ Farmer i 's profit function is then:

$$\pi_i(h_i, s_i) = ph_i - c(h_i) - \gamma s_i. \quad (7)$$

The farmers have exclusive access to the common grassland. Besides non-excludability, there is also rivalry between the farmers as livestock compete for available grass and land. If grass is taken as a fixed production factor or assumed to have simple dynamics and farmers simultaneously and independently choose animal off-take to maximize individual welfare, the negative externality imposed by one farmer on the other farmers will not be internalized. The result is over-exploitation of the grassland compared to a first best solution, where the first best solution is obtained by maximizing some social welfare function. This is the conventional explanation to over-grazing.

However, the standard common pool resource explanation to grassland degradation does not satisfactorily account for resource dynamics. To draw a more complete picture of over-grazing problems, we need to build a model that combines both building blocks — resource dynamics and strategic interactions. We call it the grazing game model.

2.3 *The grazing game model*

In the grazing game model cooperative farmers choose harvest levels to maximize the sum of their discounted joint profit, given the dynamic interaction between livestock and grass. As we assume that there are no other conflicting interests over the grassland, the externality is fully internalized if the farmers cooperate and choose harvest to maximize the sum of all farmers' profit. If farmers do not cooperate, we assume that they use open loop strategies. Hence, each farmer decides in the very beginning of the time span how to behave and commits to this strategy. In an open loop Nash equilibrium, each profit function is maximized separately, given the dynamic constraint of the grassland and the farmers' expectations about the grazing path of the other

farmers.

Let $\mathbf{s} = (s_i)_{i \in I}$ and $\mathbf{h} = (h_i)_{i \in I}$ denote the vector of livestock, and harvests respectively, $V^c(\mathbf{s}, g)$ and $V_i^{nc}(s_i, g)$, the value function for the farmers in cooperation and for farmer i in non-cooperation, respectively. Finally let Ψ^{-i} denote the expectation of farmer i of the number of animals put on the grassland by the other farmers, thus $\Psi^{-i} = \sum_{j \neq i} s_j$, where $j \in \{1, \dots, n\}$. Table 1 presents the farmers' problems, with an infinite time horizon.

Table 1: The farmers' dynamic problems

Cooperation	Non-cooperation
$V^c(\mathbf{s}, g) = \max_{\mathbf{h}} \int_0^\infty \sum_i \pi_i(h_i, s_i) e^{-\rho t} dt$	$V_i^{nc}(s_i, g) = \max_{h_i} \int_0^\infty \pi_i(h_i, s_i) e^{-\rho t} dt$
$\forall i \in I, \dot{s}_i = s_i b(g) \left(1 - \frac{\sum_i s_i}{k}\right) - h_i$	$\dot{s}_i = s_i b(g) \left(1 - \frac{s_i + \Psi^{-i}}{k}\right) - h_i$
$\dot{g} = (a + g)(1 - g) - \sum_i s_i \frac{g^2}{x^2 + g^2}$	$\dot{g} = (a + g)(1 - g) - (s_i + \Psi^{-i}) \frac{g^2}{x^2 + g^2}$

We use Pontryagin's principle (1964) to solve these dynamic problems. Let λ_i and μ_i denote the respective shadow prices for livestock and grass for farmer i . The current value Hamiltonian and the necessary conditions for a maximum for both management regimes are given in the Appendix. We obtain a modified dynamic Hamiltonian system (MDHS) with $2(n + 1)$ equations for the cooperation problem and n identical systems of four equations for the non-cooperation problem. These equation systems describe the dynamics of the managed grassland in each management regime.

We simplify the analysis and simulations considerably if we instead look at the four-equation systems, where stocks and harvests are aggregated. This is possible and

equivalent if we assume that the farmers have identical stocks and harvests at the beginning of the period. Then at each point in time, the stocks and harvests will remain identical, $s_i = \frac{s}{n}$ and $h_i = \frac{h}{n}$ for every $i \in I$. This also implies that $\lambda_i = \frac{\lambda}{n}$ and $\mu_i = \frac{\mu}{n}$ for every $i \in I$.

Because farmers control the system only through their animal off-take, the equations of motion for livestock, grass and for the shadow price of grass have identical functional forms no matter whether farmers cooperate or not:

$$\dot{s} = sb(g) \left(1 - \frac{s}{k}\right) - h, \quad (8)$$

$$\dot{g} = (a + g)(1 - g) - s \frac{g^2}{x^2 + g^2}, \quad (9)$$

$$\dot{\mu} = \mu \left(\rho - 1 + 2g + a + s \frac{2gx^2}{(x^2 + g^2)^2} \right) - (p - c'(h/n)) s \left(1 - \frac{s}{k}\right) b'(g). \quad (10)$$

However, the equations of motion for animal off-take differ. For cooperation, the harvest dynamics are given by equation (11).

$$\dot{h} = \frac{n}{c''(h/n)} \left(\left((p - c'(h/n)) \left(b(g) \left(1 - \frac{2s}{k}\right) - \rho \right) \right) - \gamma - \mu \frac{g^2}{x^2 + g^2} \right) \quad (11)$$

If farmers do not cooperate, harvest dynamics are given by equation (12).

$$\dot{h} = \frac{n}{c''(h/n)} \left((p - c'(h/n)) \left(b(g) \left(1 - \frac{(n+1)s}{nk}\right) - \rho \right) - \gamma - \frac{\mu}{n} \frac{g^2}{x^2 + g^2} \right). \quad (12)$$

Consider first the system where there is no feedback to animal performance. Such a grazing system can be viewed as one where farmers maximize profits considering only livestock dynamics, thus ignoring the livestock - grassland interaction. This implies that they attach no value to the grassland, i.e. the shadow value of grass is zero. In a system where there is a feedback to animal performance, farmers account for grassland dynamics in their maximization problem and the shadow value of grass can be either positive or negative in steady state.

For both regimes and for both systems we note that for a given grass biomass, farmers respond to economic factors as expected. For example, the relations (11) and (12) show that farmers increase the rate of animal off-take if the absolute value of the marginal cost of holding livestock decreases, the net marginal value of animal productivity increases, and if the discount rate decreases. These factors induce farmers to initially increase their stock sizes and grazing pressure, which enable them to harvest more in later periods. For both systems, the difference between the harvest dynamics stems from the fact that non-cooperative farmers only account for the crowding effects of their own stock, $s(n + 1)/n$, whereas cooperative farmers account for the crowding effect of the total stock, $2s$. In the system with feedback, also the shadow value of grazing for non-cooperative farmers is n times smaller than the corresponding value for cooperative farmers.

The next task is to analyze and compare these dynamic systems with respect to steady states and to systems dynamics. The results are presented in the following section.

3 Analysis

3.1 *Simulation methods*

We analyze the MDHSs to find possible outcomes of our grazing game, depending on whether the actors cooperate or not. We show in Appendix that for each management regime and for each way of modeling livestock growth, individual harvest, $h^*(g)$, total animal stock, $s^*(g)$, and grass shadow price, $\mu^*(g)$, are uniquely determined as functions of grass biomass. The number of steady states in a system depends then on the number of solutions to an implicit equation for each management regime where grass is the only unknown variable. This equation is highly non-linear and cannot be solved analytically so we need to rely on simulations. For a given set of parameter values¹¹ we use MATHCAD to obtain analytical expressions for the steady states.

The actual outcomes of the grazing games depend first of all on the number of steady states that satisfy necessary and sufficient conditions for local optimality.¹² If there is only one possible such state, the outcome is trivial, the optimal trajectory will lead towards this unique steady state. When several steady states satisfy the necessary and sufficient conditions for local optimality, we need to choose among them. For any candidate locally optimal trajectory we can associate a candidate value function $V = \frac{1}{\rho} \widehat{\mathcal{H}}(((s_i), g), ((\lambda_i^*), \mu^*), (h_i))$, where $\widehat{\mathcal{H}}$ denotes the maximized Hamiltonian for the problem, $\max_{h_1, h_2, \dots, h_n} \mathcal{H}(((s_i)_{i \in I}, g), ((\lambda_i^*)_{i \in I}, \mu^*), (h_i)_{i \in I})$. The global optimum for cooperative farmers starting at the initial state (s_0, g_0) , is denoted $W^c(s_0, g_0)$ and for non-cooperative farmers $W^{nc}(s_0, g_0)$. For a given regime the global optimum is obtained by comparing the values of the optimal trajectories from the starting point (s_0, g_0) to each candidate steady state (s^*, g^*) that satisfies necessary and sufficient conditions locally. In other words the optimal steady state must satisfy: $W^c(s_0, g_0) = \arg \max V^c(\dots)$ and $W^{nc}(s_0, g_0) = \arg \max V^{nc}(\dots)$ respectively.

We use MATLAB's routine for boundary value problems¹³ to simulate the trajectories of the MDHSs between given initial states and each steady state that satisfies necessary and sufficient conditions.¹⁴ This yields enough information to use the Hamilton-Jacobi equation for time autonomous problems to find optimal paths and hence outcomes. Although this routine is best suited for finite time problems we can approximate an infinite time problem by choosing a finite time horizon, which is long enough to allow the system to converge to the long run steady state, but still short enough to avoid unnecessary long computational time.¹⁵

3.2 Results

The MDHSs typically have several feasible steady states¹⁶ and the number of stable steady states depends on parameter values. For example, one regularity we observe is that grasslands with relatively high resilience — high values of x and a — only have one stable grass abundant state, whereas grasslands with relatively low resilience typically

have several stable steady states, at least one characterized by grass shortage (L) and at least one by grass abundance (H).¹⁷ To fit the purpose of the paper, henceforth we focus our analyses on examples of grasslands with low resilience, where there are at least two possible outcomes for each regime. For each system and for a given grassland and animal stock (i.e. for a given set of parameter values and initial conditions) we then proceed to compare cooperation with non-cooperation. Below we provide one example for each case but more examples can be found in Appendix (see Table 3 and Table 4).

Result 1 *For both grazing systems we have identified sets of parameters and initial conditions for which;*

- (a) a high grass steady state is a global optimum for both regimes (HH),*
- (b) a high grass steady state is a global optimum for the cooperative regime, whereas a low grass steady state is a global optimum for the non-cooperative regime (HL),*

In Figure 3 we have illustrated these two cases for the system where grass availability limits animal growth and similarly in Figure 4 where it does not. We note that the trajectories for the two systems look very similar.

In case *a*, (for both systems), it is optimal for cooperative as well as non-cooperative farmers to aim for the high grass state (HH) by decreasing grazing pressure. Farmers decrease their stocks rapidly, thereby quickly adjusting to the steady state level. Non-cooperative farmers consistently keep higher stocks than cooperative farmers and would benefit from reducing their stocks further, but due to the externality they will not. Instead each farmer realizes that if she reduces her stock she will share the associated benefits with the other farmers but bear the entire harvest cost herself. This outcome corresponds to the conventional result from common pool literature where non-cooperative farmers keep higher stocks in steady state, yielding less grass.

In case *b*, (for both systems), it is optimal for the cooperative regime to reach for the high grass state and for non-cooperative farmers to reach for the low grass state

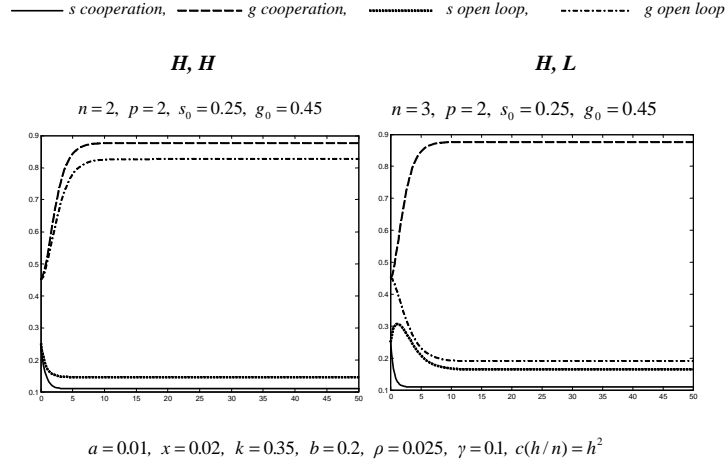


Figure 3: Cases HH and HL for a grazing system where $b(g) = b$.

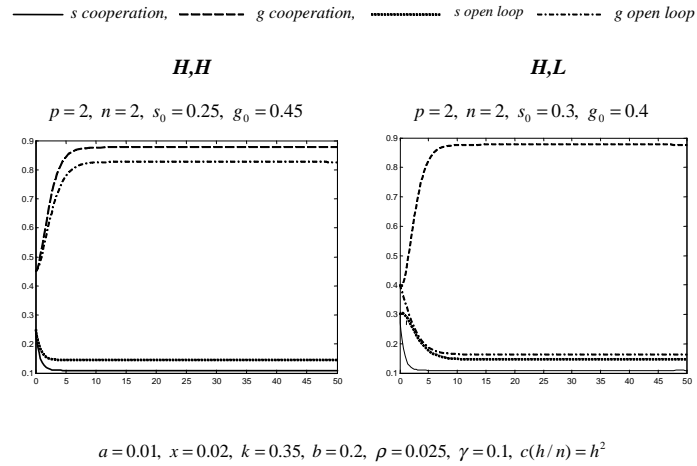


Figure 4: Cases HH and HL for a grazing system where $b(g) = bg^2/(x^2 + g^2)$.

(HL). Non-cooperative farmers do not internalize the externality of decreasing their stock sizes, which means that they under-estimate the net benefits of doing so.

When comparing case a and b we can note the following. For the system with feedback to animal performance, we increased the number of farmers keeping other parameter values as well as initial values. For the system with no feedback we changed the initial conditions slightly. Even though the changes were small in both systems, the magnitude of the externality and hence also the net benefit under-estimation were enough for the non-cooperative farmers to find it optimal to change strategy. Instead they initially increased their stock sizes, crossed the threshold, collected the revenues from harvesting and aimed for the low grass state.

From our first result we observe the following.

- The conventional negative externality associated with common property can be exacerbated due to ecosystem dynamics.

If we were to neglect the non-linearities there would only be one outcome to consider, the conventional outcome. By taking the non-linearities into account there is at least one more outcome to consider, an outcome where non-cooperative farmers exploit the grassland enough to cause an ecosystem flip. This is true no matter which of the two systems we chose to look at.

Result 2 *For a grazing system where $b(g) = b$, we have identified sets of parameters and initial conditions for which;*

- (a) a low grass steady state is a global optimum for both regimes (LL),*
- (b) a low grass steady state is a global optimum for the cooperative regime, whereas a high grass steady state is a global optimum for the non-cooperative regime (LH)*

In the first of these two cases it is optimal, for cooperative as well as non-cooperative farmers to reach for the low grass state by initially rapidly increasing grazing pressure, over-shooting the steady state stock and cross the threshold. After having crossed

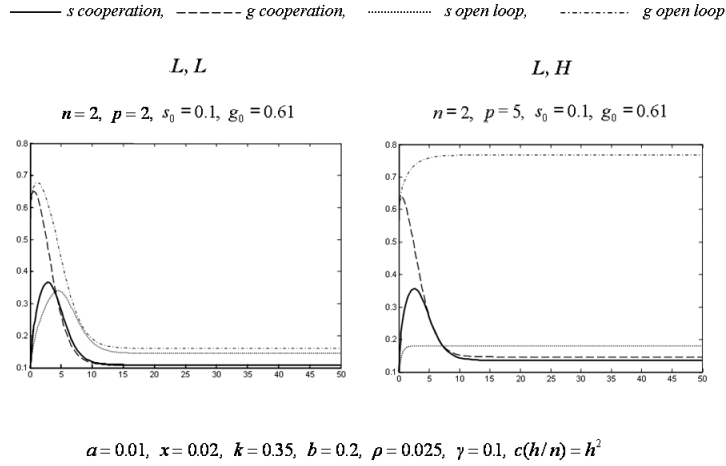


Figure 5: Cases LL and LH for a grazing system where $b(g) = b$.

threshold they decrease their stock sizes rapidly, adjusting to the steady state level. Non-cooperative farmers increase their stock sizes too little, crossing the threshold after cooperative farmers. Although non-cooperative farmers would benefit from increasing their stocks more rapidly, they will not as they do not internalize the positive externality of adding one extra unit of animal. This implies that even if stock sizes are larger, the non-cooperative regime sustains more grass in steady state. This is depicted in Figure 5.

Consider finally the case where it is optimal for the cooperative regime to reach for the low grass state and for non-cooperative farmers to reach for the high grass state (LH). Compared to case LL in Figure 5, we have only increased the price of livestock keeping other parameter values and initial values.

In this case, because non-cooperative farmers do not internalize the externality of adding on extra unit of animal, they find it too costly to increase their stocks initially. Instead they keep relatively low stock sizes and aim for the high grass state.

From our second result we note the following.

- Non-cooperative farmers may under-exploit the resource due to ecosystem dy-

namics.

The grassland is more degraded under the cooperative regime and this is true for both examples described above, although the discrepancy between non-cooperation and cooperation is larger in the latter case. We want to emphasize though that we were only able to find examples of these two cases for the system with no feedback. We also want the reader to note that for the same parameter values and initial conditions, the two systems can lead to completely different outcomes.¹⁸ These observations lead to our next remark.

- The outcome of the grazing game can be very sensitive to the model of species interaction.

In a grazing system where grass availability does not limit animal growth, farmers maximize profits accounting only for livestock dynamics (the shadow value of grass is always zero) and may end up in a high grass state or a low grass state, depending on economic factors and initial conditions. In the system where grass availability refrains animal growth, the marginal grazing effect on animal growth is higher in the low grass steady state compared to the high grass steady state. Now recall that there is not only a cost of holding livestock, but there is also a cost of keeping the stock at its steady state level in terms of harvesting. Thus, due to the feedback effect the marginal value of grass will be lower in a low grass state. As a matter of fact, when the costs of keeping the stock at its steady state outweigh the benefits from harvesting, grass becomes a bad — has a negative shadow value. This phenomenon is less pronounced when farmers do not cooperate since they over-estimate a negative shadow value of grass.¹⁹ In such a case grass is less bad for non-cooperative farmers. This probably explains why we have only been able to identify the two cases LL and LH when grass availability does not limit animal growth.

Our results and examples do not only demonstrate the qualitative influence of the non-linearities on the outcomes, however. They also show the following.

- Even marginal changes can cause radical transformations of the ecosystem.

We have consciously chosen to illustrate the different cases using practically the same set of parameter values and initial conditions, only making small changes in either one of the parameters or one of the initial conditions. Even though these changes are marginal they lead to completely different outcomes.

4 Discussion

Although grasslands are often managed as common property, it seems that this cannot alone explain their degradation. If grass production was modeled as a concave function, we would obtain the standard result from the common pool resource literature. However, we show that we do not necessarily if we account for grasslands' non-convexities. In some cases we verify the standard result but we also find situations where threshold effects magnify the conventional externality, as well as situations where non-cooperative farmers exploit the grassland less than cooperative farmers would do.

Besides non-convexities, the nature of the feedbacks involved between different variables is relevant for the results. When animal growth rate depends directly on the amount of grass available, the system has a built-in negative feedback that can refrain the animal stock from becoming too large. This is especially pronounced in a cooperative regime. When grass availability does not limit animal growth, stocks can become large enough to cause a flip to a low grass state even in a cooperative regime. Thus one possible explanation to the observed degradation of grasslands is that farmers ignore the impacts of additional animals to future forage productivity.

We show that the system is also sensitive to parameter values and initial conditions. For policy purposes, it would be interesting to identify regularities between parameter values and initial states and the corresponding outcomes. However, such a characterization would require the identification of all so-called Skiba points²⁰. (Skiba 1978). Unfortunately it is still unclear what the necessary conditions are for a Skiba

point to occur in two dimensions. (See for example Wagener, 2003 and Brock and Starrett, 2003.) In higher dimensions, like in our model, the lack of knowledge is even deeper. (Feichtinger, 2004.)

From a policy perspective, it is also important to realize that cooperative farmers will push the grassland to an alternate state if it is optimal for them to do so. Throughout the paper we have considered a case where the externalities associated with the common pool resource setting are internalized if farmers cooperate. In reality there may be other conflicting interest regarding the grassland besides those associated with grazing²¹. If so, the cooperative outcome is not the socially optimal outcome. This has some policy implications; if a social planner observes degradation of shared grasslands and wishes to restore them, she cannot conclude that a cooperative solution will solve the problem. In fact, it might be better if non-cooperative farmers managed the grasslands. Thus, assigning property rights by dividing the land may not be the best solution. The social planner may instead prefer to regulate the livestock size through quotas or taxes on livestock. However, the results from Mäler et al (2003) indicate that the effects of such policy interventions must be carefully studied to make sure that they can be used at all.

The existence of multiple steady states is coupled to grasslands' resilience. In the grassland model used, for a given system, low resilience translates into low grass re-growth potential and half saturation grass biomass. These are exogenous parameters in the model at hand but one could imagine situations in which they vary for one reason or the other. Increased drought or the introduction of invasive species, due to climate change, could for example lead to changes in grass re-growth potential. Changes in livestock composition or in the population of wild herbivores could also lead to changes in the half saturation grass biomass. In ecosystems with non-convexities, the factors influencing resilience are often slow endogenous variables or change in a stochastic way, which could affect management strategies as well. (See for example Brock & Xepapadeas (2003) for a treatment of stochastic problems with thresholds

and Crépin (2007) for a treatment of a problem with a threshold and fast and slow variables interactions.) Endogenous factors influencing grassland resilience could very well introduce a positive feedback in the social-ecological grazing system, which would probably strengthen our results and could also outweigh the negative grazing feedback effect mentioned previously.

Although we obtain our results for farmers sharing grassland, similar results could probably be derived for other kinds of shared ecosystems with complex dynamics as well. Recent findings show that such dynamics seems to characterize many resources today. (Steffen 2004.) However, our results indicate also that one must be careful and study each specific situation. One must also be sure to include sufficient amount of details and relevant information when analyzing and recommending remedies to problems like this. These observations point to the need for more general and conceptual research that can help identify key processes around threshold dynamics in social-ecological systems.

Notes

¹This expression was minted by Hardin (1968) . His example, although miss-specified, was in fact a number of herdsmen sharing a common pastureland.

²The grassland models presented here are in so called non-dimensional form. We have showed how these are equivalent to more general models. These derivations can be obtained upon request from the authors.

³For example cattle, sheep or goats.

⁴Land scarcity and competition between livestock affect livestock growth through the carrying capacity.

⁵If animal off-take is negative it simply means that the farmer acquires additional animals.

⁶Scholes and Archer (1997) and Scholes (2003) have modeled grazing systems in different ways but we chose to rely on Janssen Anderies and Walker because they actually modeled the dynamics between grass and domestic grazers.

⁷If grazing is non-linear ($x > 0$) but resilience is relatively high, $S_\theta(g)$ is strictly convex and the system only has one stable steady state. If grazing is linear, $x = 0$, $S_\theta(g)$ is strictly concave and has

one stable steady state.

⁸The motivation behind the phase diagrams can be found in appendix.

⁹This cost can be positive or negative because there is not only a maintenance cost associated with having a large stock. The stock also produces animal products and moreover, a large stock size could also serve as a measure of status. (Walker 1993.)

¹⁰The marginal cost of holding livestock is constant. The assumption of convexity of the harvest cost is crucial for ensuring an interior solution, but this is not true for the cost of holding livestock. However, the assumption definitely simplifies the derivations and presentations of the results.

¹¹We do not have the ambition to model a specific piece of grassland so we use ad hoc parameter values and check how sensitive the model is to variations.

¹²We obtained the MDHS's from a set of necessary conditions for a maximum. In general these are not sufficient. However, when certain concavity conditions are satisfied the set of necessary conditions are also sufficient locally. For our problem we have derived the concavity conditions required for sufficiency. These can be found in Appendix. We have used these concavity conditions to check that the steady states referred to in our examples are local maximum points.

¹³Shampine, L. F., Kierzenka, J., Reichelt, M. W., (2000), Solving Boundary Value Problems for Ordinary Differential Equations in MATLAB with `bvp4c`, <http://www.mathworks.com/matlabcentral/fileexchange>.

¹⁴The program used is available upon request from the authors.

¹⁵For some sets of parameter values (for details see Appendix), we simulated the model trying several different values of final time, ranging from $T=10$ to $T=1000$. From these simulations we found that the results were robust, i.e showing similar patterns and returning the same social welfare, even for rather low values of final time ($T=20$) and concluded that it would be sufficient to use $T=50$ for the remainder of the simulations.

¹⁶The general saddle path property inherent to Hamiltonian systems applies here as well: the steady states come in pairs of which one is a saddle and the other is unstable. (Birkhoff 1927 , Kurz 1968, Crépin 2002.)

¹⁷See Table 2 in Appendix.

¹⁸Compare for example the outcomes of the two systems (see Table 3 and 4 in Appendix) for the following parameter values and initial conditions; $a = 0.01, b = 0.2, \gamma = 0.1, k = 0.35, \rho = 0.025, x = 0.02, n = 3, p = 2, s_0 = 0.25, g_0 = 0.45$.

¹⁹Compare μ with μ/n in equations (11) and (12).

²⁰Starting points for which two candidate solutions that reach different steady states have the same value.

²¹Grasslands can for example be used for coal storage, fuel, recreation and tourism.

Appendix

Simulations

We simulated steady states and trajectories using many sets of parameter values and initial values. However, due to space limitations we only present a few of them here. For most of the simulations presented here, and unless otherwise stated, we used the following cost function, $c(h) = (h/n)^2$ and parameter values, $a = 0.01, b = 0.2, \gamma = 0.1, k = 0.35, \rho = 0.025, x = 0.02, n = 2, p = 2$.

Table 2. The effect of resilience on the number of steady states

	$b(g) = bg^2 / (x^2 + g^2)$				$b(g) = b$				
	low resilience				low resilience				
	s^*	g^*	h^*	μ^*	s^*	g^*	h^*	μ^*	
E_1^c	0.116	0.118	.015	-0.021	E_1^c	0.109	0.108	0.015	0
E_2^c	0.109	0.877	.015	4.5×10^{-5}	E_2^c	0.109	0.877	0.015	0
					E_3^c	0.109	0.009	0.015	0
E_1^{nc}	0.147	0.163	.017	-9.9×10^{-3}	E_1^{nc}	0.145	0.160	0.017	0
E_2^{nc}	0.145	0.826	.017	6.9×10^{-5}	E_2^{nc}	0.145	0.826	0.017	0
					E_3^{nc}	0.145	0.007	0.017	0
	high resilience; $x = 0.2$				high resilience; $x = 0.2$				
	s^*	g^*	h^*	μ^*	s^*	g^*	h^*	μ^*	
E^c	0.104	0.890	0.014	3.6×10^{-3}	E^c	0.109	0.884	0.015	0
E^{nc}	0.139	0.846	0.016	5.3×10^{-3}	E^{nc}	0.145	0.848	0.017	0
	high resilience; $a = 0.3$				high resilience; $a = 0.3$				
	s^*	g^*	h^*	μ^*	s^*	g^*	h^*	μ^*	
E^c	0.109	0.910	0.015	2.8×10^{-5}	E^c	0.109	0.910	0.015	0
E^{nc}	0.145	0.877	0.017	3.7×10^{-5}	E^{nc}	0.145	0.877	0.017	0

Table 3. Simulations for a grazing system where $b(g) = b$

$s_L^{*c} = 0.109,$		$s_H^{*c} = 0.109,$		$g_L^{*c} = 0.108,$		$g_H^{*c} = 0.877$	
$s_L^{*nc} = 0.145,$		$s_H^{*nc} = 0.145,$		$g_L^{*nc} = 0.160,$		$g_H^{*nc} = 0.826$	
s_0	g_0	h_0	μ_0	<i>outcome</i>	W^{co}	W^{nc}	
0.1	0.61	0.01	0	LL^*	2.3622	1.3149	
0.1	0.65	0.01	0	LH^*	1.6406	0.8092	
0.17	0.65	0.01	0	LL	1.4357	0.9833	
0.17	0.72	0.01	0	HH	0.8786	0.7338	
0.25	0.45	0.01	0	HH^*	1.0252	0.5218	
0.25	0.48	0.01	0	HL^*	1.0252	0.5369	
<hr/>							
$n = 3$							
$s_L^{*c} = 0.109,$		$s_H^{*c} = 0.109,$		$g_L^{*c} = 0.108,$		$g_H^{*c} = 0.877$	
$s_L^{*nc} = 0.164,$		$s_H^{*nc} = 0.164,$		$g_L^{*nc} = 0.190,$		$g_H^{*nc} = 0.797$	
s_0	g_0	h_0	μ_0	<i>outcome</i>	W^{co}	W^{nc}	
0.17	0.65	0.01	0	LL	1.5038	0.8478	
0.17	0.67	0.01	0	LH	1.3077	0.7160	
0.25	0.44	0.01	0	HH	1.0302	0.3749	
0.25	0.45	0.01	0	HL^*	1.0302	0.3777	
<hr/>							
$p = 5$							
$s_L^{*c} = 0.136,$		$s_H^{*c} = 0.136,$		$g_L^{*c} = 0.146,$		$g_H^{*c} = 0.840$	
$s_L^{*nc} = 0.181,$		$s_H^{*nc} = 0.181,$		$g_L^{*nc} = 0.220,$		$g_H^{*nc} = 0.768$	
s_0	g_0	h_0	μ_0	<i>outcome</i>	W^{co}	W^{nc}	
0.1	0.57	0.01	0	LL	7.8514	4.7675	
0.1	0.61	0.01	0	LH^*	6.4407	3.0133	

Table 4. Simulations for a grazing system where $b(g) = bg^2/(x^2 + g^2)$

$s_L^{*c} = 0.116,$		$s_H^{*c} = 0.109,$		$g_L^{*c} = 0.118,$		$g_H^{*c} = 0.877$	
$s_L^{*nc} = 0.147,$		$s_H^{*nc} = 0.145,$		$g_L^{*nc} = 0.163,$		$g_H^{*nc} = 0.826$	
s_0	g_0	h_0	μ_0	<i>outcome</i>	W^c	W^{nc}	
0.17	0.3	0.01	0	<i>HH</i>	0.8778	0.7325	
0.17	0.8	0.01	0	<i>HH*</i>	0.8779	0.7330	
0.19	0.73	0.01	0	<i>HH</i>	0.9154	0.6921	
0.19	0.76	0.01	0	<i>HH*</i>	0.9155	0.6921	
0.25	0.45	0.01	0	<i>HH*</i>	1.0245	0.5209	
0.3	0.4	0.01	0	<i>HL*</i>	1.1115	1.0062	
0.3	0.5	0.01	0	<i>HL</i>	1.1116	0.8713	
<i>n = 3</i>							
$s_L^{*c} = 0.116,$		$s_H^{*c} = 0.109,$		$g_L^{*c} = 0.118,$		$g_H^{*c} = 0.877$	
$s_L^{*nc} = 0.164,$		$s_H^{*nc} = 0.164,$		$g_L^{*nc} = 0.191,$		$g_H^{*nc} = 0.798$	
s_0	g_0	h_0	μ_0	<i>outcome</i>	W^c	W^{nc}	
0.15	0.5	0.01	0	<i>HH</i>	0.8417	0.7724	
0.17	0.7	0.01	0	<i>HH</i>	0.8803	0.7151	
0.25	0.45	0.01	0	<i>HH</i>	1.0295	0.3739	
0.3	0.38	0.01	0	<i>HL</i>	1.1189	1.1098	
0.3	0.65	0.01	0	<i>HH</i>	1.0295	0.9260	
<i>p = 5</i>							
$s_L^{*c} = 0.140,$		$s_H^{*c} = 0.136,$		$g_L^{*c} = 0.152,$		$g_H^{*c} = 0.841$	
$s_L^{*nc} = 0.182,$		$s_H^{*nc} = 0.181,$		$g_L^{*nc} = 0.222,$		$g_H^{*nc} = 0.768$	
s_0	g_0	h_0	μ_0	<i>outcome</i>	W^c	W^{nc}	
0.23	0.5	0.01	0	<i>HH</i>	3.2325	2.4376	
0.25	0.45	0.01	0	<i>HH</i>	3.3271	2.0910	

In Tables 2, 3 and 4, we denote cooperation respectively non-cooperation with suffices c and nc .

Most of the simulated results presented in Table 3 and 4 are obtained using $T=50$ as final time. However, the asterisks (*) indicate the results we also tested using $T=1000$. Note that the examples presented in the result section of the paper have all been tested using $T=1000$.

Motivation for phase diagram

Solving for total animal stock:

$$S_\phi(g) = \frac{k}{2} \left(1 \pm \sqrt{1 - \frac{4h}{bk} \frac{x^2 + g^2}{g^2}} \right) \quad (1a)$$

$$S_\varpi(g) = \frac{k}{2} \left(1 \pm \sqrt{1 - \frac{4h}{bk}} \right) \quad (1b)$$

$$S_\theta(g) = (a + g)(1 - g) \left(\frac{x^2 + g^2}{g^2} \right). \quad (1c)$$

Moreover;

$$\frac{\partial S_\phi}{\partial g} = \pm \frac{2hx^2}{bg^3 \sqrt{1 - \frac{4h}{bk} \frac{x^2 + g^2}{g^2}}}, \quad (2)$$

$$\frac{\partial S_\varpi}{\partial g} = 0, \quad (3)$$

$$\frac{\partial S_\theta}{\partial g} = -\frac{2(g^4 + ax^2) - (1 - a)(g^2 - x^2)g}{g^3}. \quad (4)$$

So $S_\phi(g)$ has an increasing upper part and a decreasing lower part and $S_\varpi(g)$ is constant for all grass biomasses. Note that $\lim_{g \rightarrow 0} S_\phi(g, a, b, x)$ is undefined and that $S_\phi(1, a, b, x) = 2hx^2/b\sqrt{1 - \frac{4h}{bk}x^2}$.

The sign of the derivative $\frac{\partial S_\theta}{\partial g}$ depends on parameter values. More specifically, the derivative is negative if $2(g^4 + ax^2) - (1 - a)(g^2 - x^2)g > 0$. Let $\alpha(g) = 2(g^4 + ax^2)$ and $\beta(g) = (1 - a)(g^2 - x^2)g$. It is easily verified that $\alpha(g)$ is increasing and convex for any positive value of a and x , and $\alpha(0) = 2ax^2$. For $\beta(g)$ we have that

$$\frac{\partial \beta}{\partial g} = (3g^2 - x^2)(1 - a), \text{ and } \frac{\partial^2 \beta}{\partial g^2} = 6g(1 - a). \quad (5)$$

The graph of β depends on parameter values. But if we follow Janssen, Anderies and Walker (2004), the re-growth potential, a should be lower than 1. This implies that β has a minimum at $g = \frac{\sqrt{3}}{3}x \equiv \check{g}$ and is convex. Moreover, $\beta(0) = \beta(x) = 0$ and $\beta(\check{g}) = -\frac{2}{9}(1-a)x^3\sqrt{3}$. From this, we can conclude that α and β can intersect if x is sufficiently small compared to a . This means that there will be an interval on which $\frac{ds}{dg}\Big|_{g=0}$ is positive for intermediate values of g . If grass biomass is small or if grass biomass is large, $\frac{ds}{dg}\Big|_{g=0}$ is negative. If x is large compared to a , the curves do not intersect and $\frac{dS}{dg}\Big|_{g=0} < 0$ for any grass biomass level. Note also that $\lim_{g \rightarrow 0} S_\theta(g, a, b, x) = +\infty$, and $S_\theta(1, a, b, x) = 0$.

Pontryagin's principle for cooperation

$$V^c(\mathbf{s}, g) = \max_{\mathbf{h}} \int_0^{+\infty} \sum_{i=1}^n (ph_i - c_1(h_i) - \gamma s_i) e^{-\rho t} dt \quad (6)$$

st :

$$\dot{s}_1 = s_1 b(g) \left(1 - \frac{\sum_i s_i}{k}\right) - h_1, \dots, \quad (7)$$

$$\dot{s}_n = s_n b(g) \left(1 - \frac{\sum_i s_i}{k}\right) - h_n \quad (8)$$

$$\dot{g} = (a + g)(1 - g) - \sum_{i=1}^n s_i \frac{g^2}{x^2 + g^2}. \quad (9)$$

$\mathcal{H}(s, g, h, \lambda, \mu)$ denotes the Hamiltonian

$$\begin{aligned} \mathcal{H}(s_i, g, h_i, \lambda_i, \mu) &= \sum_{i=1}^n (ph_i - c(h_i) - \gamma s_i) \\ &+ \sum_{i=1}^n \lambda_i \left(s_i b(g) \left(1 - \frac{\sum_i s_i}{k} \right) - h_i \right) \\ &+ \sum_{i=1}^n \mu_i \left((a + g)(1 - g) - \sum_i s_i \frac{g^2}{x^2 + g^2} \right) \end{aligned} \quad (10)$$

Assuming a unique interior solution equations (11), (12) and (13) give the first-order necessary conditions for a maximum (Seierstad and Sydsæter 1987, theorem 2 and 12).

$$\forall i \in I : p - c'(h_i) - \lambda_i = 0, \quad (11)$$

$$\forall i \in I : \dot{\lambda}_i = \lambda_i \rho + \gamma - \lambda_i b(g) \left(1 - \frac{2s_i + \sum_{j \neq i} s_j}{k} \right) + b(g) \sum_{j \neq i} \lambda_j \frac{s_j}{k} + \sum_{i=1}^n \mu_i \frac{g^2}{x^2 + g^2}, \quad (12)$$

$$\sum_{i=1}^n \dot{\mu}_i = \sum_{i=1}^n \mu_i \left(\rho - 1 + 2g + a + \sum_{i=1}^n s_i \frac{2gx^2}{(x^2 + g^2)^2} \right) - \sum_{i=1}^n \lambda_i \left(s_i b'(g) \left(1 - \frac{\sum_{i=1}^n s_i}{k} \right) \right) \quad (13)$$

The $\dot{\lambda}_i$ equation transforms into:

$$\forall i \in I : \dot{\lambda}_i = \lambda_i \rho - b(g) \left(\lambda_i \left(1 - \frac{2s_i + \sum_{j \neq i} s_j}{k} \right) - \sum_{j \neq i} \lambda_j \frac{s_j}{k} \right) + \gamma + \sum_{i=1}^n \mu_i \frac{g^2}{x^2 + g^2}, \quad (14)$$

These conditions are also sufficient if the concavity conditions are satisfied and for all admissible $\mathbf{j}(t) = (s_1, \dots, s_n, g)'$:

$\lim_{t \rightarrow +\infty} (\lambda_1, \dots, \lambda_n, \mu)(\mathbf{j}(t) - \mathbf{j}^*(t)) \geq 0$. (Seierstad and Sydsæter 1987, theorems 5 and 14). Equation (11) gives $\dot{\lambda}_i = -c_1''(h_i) \dot{h}_i$ and can be used together with (12) to

obtain an equation of motion for animal off-take.

$$\dot{h}_i = \frac{1}{c_1'(h_i)} \left((p - c'(h_i)) \left(b(g) \left(1 - 2 \frac{\sum_i s_i}{k} \right) - \rho \right) - \gamma - \sum_{i=1}^n \mu_i \frac{g^2}{x^2 + g^2} \right)$$

Consider instead a four-equation system, where stocks, harvests and the shadow prices for grass are aggregated. This is equivalent to the system above because the farmers are symmetric and start with identical stocks and harvests; i.e. $s_i = \frac{s}{n}$, $h_i = \frac{h}{n}$ and $\mu_i = \frac{\mu}{n}$ for every $i \in I$.²²

$$\dot{s} = sb(g) \left(1 - \frac{s}{k} \right) - h, \quad (15)$$

$$\dot{g} = (a + g)(1 - g) - s \frac{g^2}{x^2 + g^2}, \quad (16)$$

$$\dot{h} = \frac{n}{c_1'(h/n)} \left((p - c'(h/n)) \left(b(g) \left(1 - \frac{2s}{k} \right) - \rho \right) - \gamma - \mu \frac{g^2}{x^2 + g^2} \right) \quad (17)$$

$$\dot{\mu} = \mu \left(\rho - 1 + 2g + a + s \frac{2gx^2}{(x^2 + g^2)^2} \right) - (p - c'(h/n)) s \left(1 - \frac{s}{k} \right) b'(g) \quad (18)$$

Pontryagin's principle in non-cooperation.

$$V_i^{nc}(s_i, g) = \max_{h_i} \int_0^{+\infty} (ph_i - c(h_i) - \gamma s_i) e^{-\rho t} dt \quad (19)$$

s.t.:

$$\dot{s}_i = s_i b(g) \left(1 - \frac{s_i + \Psi^{-i}}{k} \right) - h_i, \quad (20)$$

$$\dot{g} = (a + g)(1 - g) - (s_i + \Psi^{-i}) \frac{g^2}{x^2 + g^2}. \quad (21)$$

$$\begin{aligned}
\mathcal{H}_i^{nc}(s_i, h_i, g, \lambda_i, \mu) &= ph_i - c(h_i) - \gamma s_i \\
&+ \lambda_i \left(s_i b(g) \left(1 - \frac{s_i + \Psi^{-i}}{k} \right) - h_i \right) \\
&+ \mu_i \left((a + g)(1 - g) - (s_i + \Psi^{-i}) \frac{g^2}{x^2 + g^2} \right). \quad (22)
\end{aligned}$$

Assuming a unique interior solution, equations (23), (24) and (25) give the first-order necessary conditions for a maximum,

$$p - c'(h_i) - \lambda_i = 0, \quad (23)$$

$$\dot{\lambda}_i = \lambda_i \rho - \lambda_i \left(b(g) \left(1 - \frac{2s_i + \Psi^{-i}}{k} \right) \right) + \gamma + \mu_i \frac{g^2}{x^2 + g^2} \quad (24)$$

$$\dot{\mu}_i = \mu_i \left(\rho - 1 + 2g + a + (s_i + \Psi^{-i}) \frac{2gx^2}{(x^2 + g^2)^2} \right) - \lambda_i s_i \left(1 - \frac{s_i + \Psi^{-i}}{k} \right) b'(g) \quad (25)$$

Applying the same technique used for the cooperative case, we once more obtain an equation of motion for harvest

$$\dot{h}_i = \frac{1}{c_1''(h_i)} \left((p - c'(h_i)) \left(b(g) \left(1 - \frac{2s_i + \Psi^{-i}}{k} \right) - \rho \right) - \gamma - \mu_i \frac{g^2}{x^2 + g^2} \right) \quad (26)$$

Similar to the cooperation case, we simplify the analysis and the simulations considerably if we look at the four-equation system. Moreover, we let $\Psi^{-i} = (n - 1) \frac{s}{n}$,

then

$$\dot{s} = sb(g) \left(1 - \frac{s}{k}\right) - h, \quad (27)$$

$$\dot{g} = (a + g)(1 - g) - s \frac{g^2}{x^2 + g^2}, \quad (28)$$

$$\dot{h} = \frac{n}{c'_1(h/n)} \left((p - c'(h/n)) \left(b(g) \left(1 - \frac{(n+1)s}{nk}\right) - \rho \right) - \gamma - \frac{\mu}{n} \frac{g^2}{x^2 + g^2} \right) \quad (29)$$

$$\dot{\mu} = \mu \left(\rho - 1 + 2g + a + s \frac{2gx^2}{(x^2 + g^2)^2} \right) - (p - c'(h/n)) s \left(1 - \frac{s}{k}\right) b'(g) \quad (30)$$

Number of interior steady states

Recall the MDHS in cooperation and non-cooperation, i.e. equations (15), (16), (18), (17) (cooperation) and (29) (non-cooperation). So in steady state we have the following relations (assuming $\rho - 1 + 2g + a + s \frac{2gx^2}{(x^2 + g^2)^2} \neq 0$) for the system with a negative feedback

$$h^*(g) = bs^*(g) \frac{g^2}{x^2 + g^2} \left(1 - \frac{s^*(g)}{k}\right) \quad (31)$$

$$s^*(g) = (a + g)(1 - g) \frac{x^2 + g^2}{g^2} \quad (32)$$

$$\mu^*(g) = \frac{\left(p - c'\left(\frac{h^*(g)}{n}\right)\right) s^*(g) b\left(1 - \frac{s^*(g)}{k}\right) 2gx^2}{(x^2 + g^2)^2 (\rho - 1 + 2g + a) + 2s^*(g) gx^2} \quad (33)$$

where we again omit to separately specify the function for both regimes. From equations (31) and (32) we can derive the following expression for harvest, and shadow price of grass as function of grass biomass.

$$h^*(g) = b(a+g)(1-g) \left(1 - \frac{(a+g)(1-g)(x^2+g^2)}{kg^2} \right) \quad (34)$$

$$\mu^*(g) = \frac{\left(p - c' \left(\frac{h^*(g)}{n} \right) \right) (a+g)(1-g) b (kg^2 - (a+g)(1-g)(x^2+g^2)) 2x^2}{((x^2+g^2)(\rho-1+2g+a)g + 2(a+g)(1-g)x^2) kg^2} \quad (35)$$

Grass biomass differs in both regimes and is determined as the solution to equation (36) in cooperation and (37) in non-cooperation

$$\left(p - c' \left(\frac{h^*(g)}{n} \right) \right) \left(\frac{bg^2}{x^2+g^2} - \frac{b2(a+g)(1-g)}{k} - \rho \right) - \gamma - \mu^*(g) \frac{g^2}{x^2+g^2} = 0 \quad (36)$$

$$\left(p - c' \left(\frac{h^*(g)}{n} \right) \right) \left(\frac{bg^2}{x^2+g^2} - \frac{b(n+1)(a+g)(1-g)}{nk} - \rho \right) - \gamma - \frac{\mu^*(g)}{n} \frac{g^2}{x^2+g^2} = 0 \quad (37)$$

which can be written as:

$$b \left(\frac{g^2}{x^2+g^2} - \frac{2(a+g)(1-g)}{k} \right) - \rho - \frac{\gamma}{(p - c'(h^*(g)/n))} = \frac{(a+g)(1-g) b (kg^2 - (a+g)(1-g)(x^2+g^2)) 2x^2}{((x^2+g^2)(\rho-1+2g+a)g + 2(a+g)(1-g)x^2) k} \frac{1}{x^2+g^2} \quad (38)$$

and

$$b \left(\frac{g^2}{x^2+g^2} - \frac{(n+1)(a+g)(1-g)}{nk} \right) - \rho - \frac{\gamma}{p - c'(h^*(g)/n)} = \frac{(a+g)(1-g) b (kg^2 - (a+g)(1-g)(x^2+g^2)) 2x^2}{n((x^2+g^2)(\rho-1+2g+a)g + 2(a+g)(1-g)x^2) k} \frac{1}{x^2+g^2} \quad (39)$$

For a system without the negative dampening feedback effect we obtain the following. Again assuming $\rho - 1 + 2g + a + s \frac{2gx^2}{(x^2+g^2)^2} \neq 0$.

$$h^*(g) = s^*(g) b \left(1 - \frac{s^*(g)}{k} \right) \quad (40)$$

$$s^*(g) = (a+g)(1-g) \frac{x^2+g^2}{g^2} \quad (41)$$

$$\mu^*(g) = 0 \quad (42)$$

From equations (31) and (32) we can derive the following expression for harvest, and shadow price of grass as function of grass biomass.

$$h^*(g) = (a+g)(1-g) b \left(1 - \frac{(a+g)(1-g)(x^2+g^2)}{kg^2} \right) \frac{x^2+g^2}{g^2} \quad (43)$$

Grass biomass differs in both regimes and is determined as the solution to equation (36) in cooperation and (37) in non-cooperation

$$\left(p - c' \left(\frac{h^*(g)}{n} \right) \right) \left(b \left(1 - \frac{2(a+g)(1-g)(x^2+g^2)}{kg^2} \right) - \rho \right) - \gamma = 0 \quad (44)$$

$$\left(p - c' \left(\frac{h^*(g)}{n} \right) \right) \left(b \left(1 - \frac{(n+1)(a+g)(1-g)(x^2+g^2)}{nkg^2} \right) - \rho \right) - \gamma = 0 \quad (45)$$

Concavity conditions

For the cooperative case the maximized Hamiltonian, $\hat{H}(s, g, \lambda, \mu) \equiv$

$\max \{\mathcal{H}(s, g, h, \lambda, \mu) : h\}$ is concave in s and g if $(-1)^r \Delta_r \geq 0$ for $r = 1, 2$, where Δ_r are principal minors of the order r in the Hessian for $\hat{H}(s, g, \lambda, \mu)$ (see Nikaido 1968). The Hessian matrix is here $\begin{pmatrix} \hat{H}_{ss} & \hat{H}_{sg} \\ \hat{H}_{sg} & \hat{H}_{gg} \end{pmatrix}$, which means that there are two minors. Thus the maximized Hamiltonian is concave in s and g if;

$$\begin{aligned} (-1)^1 \hat{H}_{ss} &\geq 0 \\ (-1)^2 \left(\hat{H}_{ss} \hat{H}_{gg} - \hat{H}_{sg} \hat{H}_{sg} \right) &\geq 0 \end{aligned}$$

We have that $p - c'(h_i^*) = \lambda_i$ so $h_i^* = \xi(p - \lambda_i)$, where ξ is the inverse function of c' .

This gives

$$\begin{aligned} \hat{H}(s_i, g, \lambda_i, \mu) &= \sum_{i=1}^n (p\xi(p - \lambda_i) - c(\xi(p - \lambda_i)) - \gamma s_i) \\ &\quad + \sum_{i=1}^n \lambda_i \left(s_i b(g) \left(1 - \frac{\sum_i s_i}{k} \right) - \xi(p - \lambda_i) \right) \\ &\quad + \sum_{i=1}^n \mu_i \left((a + g)(1 - g) - \sum_i s_i \frac{g^2}{x^2 + g^2} \right) \end{aligned} \quad (46)$$

and due to symmetry

$$\begin{aligned} \hat{H}(s_i, g, \lambda_i, \mu) &= \hat{H}(s, g, \lambda, \mu) = np(\xi(p - \lambda_i)) - nc(\xi(p - \lambda_i)) - \gamma s + \\ &\quad \lambda_i \left(sb(g) \left(1 - \frac{s}{k} \right) - n(\xi(p - \lambda_i)) \right) \\ &\quad + \mu \left((a + g)(1 - g) - s \frac{g^2}{x^2 + g^2} \right), \end{aligned} \quad (47)$$

Thus

$$\hat{H}_{ss} = -\frac{2\lambda_i b(g)}{k} < 0 \quad (48)$$

$$\hat{H}_{sg} = \hat{H}_{gs} = \lambda_i b'(g) \left(1 - \frac{2s}{k}\right) - \mu \frac{2gx^2}{(x^2 + g^2)^2} \quad (49)$$

$$\hat{H}_{gg} = \lambda_i \left(sb''(g) \left(1 - \frac{s}{k}\right) \right) - \mu s \frac{2(x^2 - 3g^2)x^2}{(x^2 + g^2)^3} - 2\mu. \quad (50)$$

A concave Hamiltonian requires;

$$\begin{aligned} \lambda_i &\geq 0 \\ -\frac{2\lambda_i b(g)}{k} &\left(\lambda_i \left(sb''(g) \left(1 - \frac{s}{k}\right) \right) - \mu s \frac{2(x^2 - 3g^2)x^2}{(x^2 + g^2)^3} - 2\mu \right) \\ &- \left(\lambda_i b'(g) \left(1 - \frac{2s}{k}\right) - \mu \frac{2gx^2}{(x^2 + g^2)^2} \right)^2 \geq 0 \end{aligned}$$

For the non-cooperative case the maximized Hamiltonian is:

$$\begin{aligned} \hat{H}_i(s_i, g_i, \lambda_i, \mu) &= p\xi(p - \lambda_i) - c(\xi(p - \lambda_i)) - \gamma s_i + \\ &\lambda_i \left(s_i b(g) \left(1 - \frac{s_i + \Psi^{-i}}{k}\right) - \xi(p - \lambda_i) \right) \\ &+ \mu_i \left((a + g)(1 - g) - (s_i + \Psi^{-i}) \frac{g^2}{x^2 + g^2} \right), \quad (51) \end{aligned}$$

the corresponding Hessian is $\begin{pmatrix} \hat{H}_{i s_i s_i} & \hat{H}_{i s_i g} \\ \hat{H}_{i s_i g} & \hat{H}_{i g g} \end{pmatrix}$, where

$$\hat{H}_{i s_i s_i} = -\frac{2\lambda_i b(g)}{k} < 0 \quad (52)$$

$$\hat{H}_{i s_i g} = \hat{H}_{i s_i g} = \lambda_i b'(g) \left(1 - \frac{2s_i + \Psi^{-i}}{k}\right) - \mu_i \frac{2gx^2}{(x^2 + g^2)^2} \quad (53)$$

$$\hat{H}_{i gg} = \lambda_i s_i b''(g) \left(1 - \frac{s_i + \Psi^{-i}}{k}\right) - \mu_i (s_i + \Psi^{-i}) \frac{(x^2 - 3g^2)x^2}{(x^2 + g^2)^3} - 2\mu_i. \quad (54)$$

A concave Hamiltonian requires;

$$\lambda_i \geq 0 \quad (55)$$

$$\left(\frac{-2\lambda_i b(g)}{k}\right) \left(\lambda_i s_i b''(g) \left(1 - \frac{s_i + \Psi^{-i}}{k}\right) - \mu_i (s_i + \Psi^{-i}) \frac{(x^2 - 3g^2)x^2}{(x^2 + g^2)^3} - 2\mu_i\right) \quad (56)$$

$$- \left(\lambda_i b'(g) \left(1 - \frac{2s_i + \Psi^{-i}}{k}\right) - \mu_i \frac{2gx^2}{(x^2 + g^2)^2}\right)^2 \geq 0, \quad (57)$$

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