The Beijer Institute of Ecological Economics

DISCUSSION PAPER

Beijer Discussion Paper Series No. 235

Structural change in a two-sector model of the climate and the economy.

Gustav Engström. 2012.

The Beijer Institute of Ecological Economics
The Royal Swedish Academy of Sciences
Box 50005 SE-104 05 Stockholm Sweden
Phone: +46 8 673 95 00 Fax: +46 8 15 24 64
E-mail: beijer@beijer.kva.se

Structural change in a two-sector model of the climate and the economy

Gustav Engström

May 15, 2012

Abstract

This paper introduces issues concerning substitution possibilities among goods into a two-sector macroeconomic growth model where emissions from fossil fuels give rise to a climate externality. Substitution possibilities are modeled using a constant elasticity of substitution (CES) production function where the intermediate inputs differ only in their technologies and the way they are affected by the climate externality. By solving the social planners problem and characterizing the competitive equilibrium I am able to derive a simple formula for optimal taxes and resource allocation over time. The impact of different assumptions regarding the elasticity of substitution on taxes turns out to be a simple function of the size or relative magnitude of the distribution parameter of the CES function, technology and the impact of the climate externality. In particular, it is shown that a higher (lower) elasticity of substitution will result in a higher (lower) optimal unit tax rate *if and only if* the distribution parameter of the most productive sector, multiplied by its total factor productivity and climate damage function, is smaller (larger) than the corresponding term of the other sector. I also present some numerical simulations for a calibrated model based on the U.S. and Indian economy. The results show that the assumptions regarding substitution possibilities plays a much bigger role for optimal fossil fuel consumption in the agriculturally intense Indian economy.

1 Introduction

In undergraduate courses in economics we learn to classify goods as being either substitutes or complements in order to determine their effect on demand, supply and prices in the market place. When goods are complements an increased scarcity in one good will increase its price relative to other goods and vice versa. In macro economics these differences among goods or factor inputs has shown to be of importance in explaining trends in the movements of capital and labor across different sectors of the economy over time (structural change). This approach to modeling structural change was originally proposed by Baumol (1967) and has recently been developed further by e.g. Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008). These later studies show that both productivity and capital intensity differences among sectors can help explain the post-industrial flow of capital and labor from the agricultural sector into the manufacturing and service sectors.[1]

Within environmental economics these properties has also been receiving attention. Recent studies by (Hoel and Sterner, 2007; Sterner and Persson, 2008), shows that when

[1] Acemoglu and Guerrieri (2008) do not attempt to explain the flow of capital and labor from agriculture into manufacturing, however later unpublished work by Lin and Xu (2011) show that this could have been explained within the context of their model.

assumptions regarding perfect substitutability between goods are relaxed this can have potentially large effects on optimal mitigation policies in climate economy models. Standard, economic models featuring a climate externality typically ignore these effects. An implicit assumption embedded in these aggregate models is thus that both consumption goods and intermediate inputs to production are perfect substitutes.[2] The paper by Sterner and Persson (2008) experimented with the well-known DICE model developed by (Nordhaus, 2007), showing that if an alternative environmental good is introduced into this model this can result in a dramatic shift in the optimal emission policy. This was done by replacing the representative consumption good of the DICE model with a composite good consisting of an environmental good and a manufactured good, which are weighted together using a constant elasticity of substitution (CES) utility function. The two intermediate goods were further assumed to be complements in the utility function implying a elasticity of substitution below unity. Since the investment decisions resulting from model simulation implied that the manufactured good was growing over time while the environmental good was becoming increasingly scarce the uneven growth rates together with a CES smaller than one, lead to a rising relative price of the environmental good. The result of this imbalance among growth rates was thus an increase in cost of climate change and hence a more stringent emission policy.[3]

In this paper I continue this line of research exploring how assumptions regarding substitutability among input factors might affect mitigation policies by developing a two-sector general equilibrium model of the climate-economy interaction. The purpose of this exercise is to extend the insights of (Hoel and Sterner, 2007; Sterner and Persson, 2008) into a more tractable general equilibrium model of the macro economy. I differ from them in three important aspects; First, I have replaced the environmental good by an agricultural sector. This is an important first step in making the model more accessible to macro economic researchers since it allows for calibration of the model based on actual macroeconomic data. The alteration of the DICE model by (Sterner and Persson, 2008) differs here since their definition of the environmental good is a much more abstract and difficult concept to find empirical data on. Further, I believe that the agricultural sector can work as a good proxy since this sector is highly dependent upon the surrounding environment such as temperature and precipitation. Second, I allow for endogenous and free mobility of resources between the two sectors. By doing so I follow in the tradition of a vast literature on multi-sector growth models. Within the climate-economy framework considered here this assumption also has a useful interpretation in terms of adaptation costs to climate change. Here, we can think of resources flowing into the most heavily damaged sector as the opportunity cost of mitigation, implying that there exists a trade off between mitigation and adaptation decisions within the model. Finally, I model substitution decisions as a supply side phenomena i.e. I look at substitution among intermediate inputs in final output. This is perhaps more of a technical aspect that increases the analytical tractability of our model. However, as will be shown the equations governing structural change are identical to those of Ngai and Pissarides (2007) when the climate externality is ignored. Further, to my knowledge this has yet not been applied within a climate-economy model.

The model developed here draws upon work by Acemoglu and Guerrieri (2008) which highlights a supply side reason for structural change based on the thesis presented by Baumol (1967). They develop a two-sector model, with a constant elasticity of substitution and show that if either capital shares or productivity differs between the two sectors

[2] Examples of such aggregate models can be found in a recent review by Stanton *et al.* (2009).

[3] Weitzman (2010) shows that under their specific assumptions regarding the elasticity of substitution this specification becomes equivalent to introducing an additive damage function affecting utility directly.

structural change will take place. Further, if the two sectors are complements in production then this implies that resources will be allocated towards the smaller of the two sectors. Our paper also draws upon work by Golosov *et al.* (2011) which develop a stochastic dynamic general-equilibrium of the climate and the economy. They show that given four specific assumptions i) logarithmic utility, ii) climate damages being proportional to output iii) the stock of atmospheric carbon dioxide grows linearly in emissions and iv) a constant saving rate, it is possible to derive a simple formula for the marginal externality cost from the emissions of carbon dioxide. These assumptions also turn out to be particularly useful for deriving analytical results in our two-sector setting.

The numerical section of this paper concludes with a simple calibration and simulation exercise. Here, I calibrate and simulate the model based on data from the U.S. and Indian economy separately. Already in the seminal article by Arrow *et al.* (1961) it was pointed out that systematic inter-sectoral differences in the elasticity of substitution and income elasticities of demand, imply the possibility that the process of economic development itself might shift the over-all elasticity of substitution. It has also for a long time been a well recognized stylized fact that as a country moves out of poverty and economic growth starts to take of, the relative economic importance of the agricultural sector starts to decline (see e.g. Timmer (2009)). Hence, since these two economies differ to a great extent in the size of their agricultural sector calibrating to their respective observed economies shows off some important differences that can prove to be of relevance when considering global optimal emission policies from the perspective of different nations or economic systems. The results show that the optimal global emission policy from the perspective of the Indian economy exhibits a more stringent emission path and is more sensitive to changes in substitution possibilities than the corresponding U.S. economy.

This paper is structured as follows. Section 2 introduces the general features of the model, derives the planner and corresponding competitive equilibrium solution. Section 3 provides some numerical calibration and simulation results. Section 4 concludes.

2 A two-sector intermediate goods model of optimal resource use

In this section the general setting and description of the planning problem and competitive equilibrium of the two-sector model is introduced. The model I develop here is a discrete time version of the model developed in Acemoglu and Guerrieri (2008) extending it to include a climate externality and fossil fuel use. In order to get the analytical results derived in this paper I will make some specific assumptions that although not completely implausible still might be regarded as overly stylized. The reason for this is related to the purpose of this paper which is to clarify the mechanism played by the elasticity of substitution in determining optimal fossil fuel use and taxes within a macroeconomic growth model. Finally, I work out the solution to the planner problem and show how this solution can be implemented in a decentralized setting given an externality correcting taxation policy.

2.1 Model description

The objective function of a representative household in the economy is given by

$$\sum_{t=0}^{\infty} \beta^t U(C_t) \tag{1}$$

where U is a standard concave the utility function function, C consumption and $\beta \in (0,1)$ is the discount factor.

The economy produces a unique final good which can be thought of as an aggregate/composite good consisting of the two intermediate goods

$$Y_t = \left(w_m Y_{mt}^{(\epsilon-1)/\epsilon} + w_a Y_{at}^{(\epsilon-1)/\epsilon}\right)^{\epsilon/(\epsilon-1)} \quad (2)$$

having a elasticity of substitution $\epsilon \in [0,\infty)$ and a distribution parameters (sectoral weights) $w_m \geq 0; w_{at} \geq 0$ and $w_m + w_a = 1$.[4] The economy is thus divided into two sectors. First, the agricultural sector Y_{at} is a proxy for all types of food related production activities. Second, manufacturing production Y_{mt} refers to all other types of production activities that do not fit into agricultural production (i.e. everything else). Both production technologies employ standard production factors such as capital K, labor L and energy E. Production functions are further assumed to be of Cobb-Douglas type with differing technological trends A_{at} and A_{mt}. Finally both sectors are also assumed to be affected differently by climate change in a multiplicative fashion.[5]

$$Y_{at} = \Omega_a(S_t) A_{at} K_{at}^{\alpha_1^a} L_{at}^{\alpha_2^a} E_{at}^{\alpha_3^a} \quad (3)$$
$$Y_{mt} = \Omega_m(S_t) A_{mt} K_{mt}^{\alpha_1^m} L_{mt}^{\alpha_2^m} E_{mt}^{\alpha_3^m} \quad (4)$$

where $\Omega_i(S_t) \in [0,1]$, A_{it}, K_{it}, L_{it} and E_{it} are the damage function associated with atmospheric carbon dioxide concentration $S_t > 0$, technological growth, capital, labor and energy use in each sector $i = \{a, m\}$ respectively. Note, that the damage function I consider here is a direct function of the atmospheric carbon dioxide stock meaning that I have surpassed several possibly important dynamical relationships such as for example ocean heating etc common in many integrated assessment models. Golosov *et al.* (2011) argue that this is a reasonable assumption given the available intermediate complexity models used in natural sciences. Although, I do not aim to take a stand here this reduced complexity makes it easier to understand the forces of driving the model we consider here.[6] Further, I will throughout this paper assume that damages are always increasing in the atmospheric carbon stock i.e. $\Omega_i^{'}(S_t) < 0$.

Finally, the economy's budget constraint in final good production is

$$K_{t+1} + C_t = Y_t + (1-\delta)K_t \quad (5)$$

where the left hand side denotes next periods resource use (capital and consumption) while the right hand side denotes production and depreciation of capital.

Regarding fossil fuel use dynamics let R_t denote the stock of remaining fossil fuel at the beginning of time period t, where $R_0 > 0$ is given, and $E_t \geq 0$ denotes the total amount of extracted fossil fuel by the two sectors.

$$R_{t+1} - R_t = -E_t, \; R(0) = R_0 > 0 \quad (6)$$

[4]For the cobb-douglas case where $\epsilon = 1$ these distribution parameters can be interpreted as the income shares of the intermediate goods in final good production.

[5]Hassler *et al.* (2011) point out that, on shorter time horizons, Cobb-Douglas production does not represent a good way of modeling energy demand since it does not capture the joint shorter- to medium-run movements of input prices and input shares. However, on longer time scale we consider here it is more reasonable since input shares do not appear to trend over time.

[6]In the numerical section of this paper I will make a simple logarthmic transformation from carbon dioxide to temperature units found in e.g. IPCC (2001).

the following resource constraint thus applies:

$$R_0 \geq \sum_{t=0}^{\infty} E_t \tag{7}$$

Capital, Labor and Energy can be allocated costlessly across both sectors. Market clearing thus requires that

$$K_t = K_{at} + K_{mt} \tag{8a}$$
$$L_t = L_{at} + L_{mt} \tag{8b}$$
$$E_t = E_{mt} + E_{at} \tag{8c}$$

Finally, I let S_t denote the stock of carbon dioxide emitted after the pre-industrial period and assume the following simple structure for the carbon cycle.

$$S_{t+1} = (1 - \varphi)S_t + \xi E_t \tag{9}$$

This equation is a much simplified expression for the behavior of anthropogenic induced CO_2 emissions following early work on climate economy models (see e.g. Nordhaus (1994)) where φ captures the rate of removal of CO_2 from the atmosphere and ξ the airborne fraction of carbon dioxide emissions. Removal might be due to for example uptake by oceans or the terrestrial biosphere. This is a rather simple an crude way of capturing carbon storage which ignores several possibly important dynamical relationships present in for example Nordhaus and Boyer (2000). However, for the purposes of the present paper these dynamics serve us well as a simplified representation. Further, as a reference Golosov *et al.* (2011) argue that increased complexity of the three box carbon cycle used by Nordhaus is quite well approximated by a simple one-dimensional lag structure.

2.2 The Planning problem

Based on the formulations described above I can now form a social planner problem and characterize a solution. The planner problem becomes

$$\max_{\{K_{t+1},R_{t+1},S_{t+1}E_t,C_t,K_{mt},K_{at},L_{at},L_{mt},E_{mt},E_{at}\}} \sum_{t=0}^{\infty} \beta^t U(C_t) \tag{10}$$

$$\text{subject to (2), (3) (4), (5), (6), (8a), (8b), (8c), (9)} \tag{11}$$

Inspection of the social planner problem reveals that this maximization problem can be broken down into two parts. First, given the state variables K_t, R_t and S_t the allocation of factors across the two sectors becomes an intratemporal problem of maximizing the aggregate output Y_t in each time period. Second, given this choice of factor allocation in each time period the time path of C_t and E_t can be chosen so as to maximize the value of the objective function. These two parts thus corresponds to the solutions of the static and dynamic maximization problems. I start by characterizing the static equilibrium.

2.2.1 Static equilibrium

As mentioned previously, in order to obtain a tractable model in terms of analytical results I will have to make some rather specific assumptions. The first assumption relates to the factor input shares within the two sectors

Assumption 1 $\alpha_j^a = \alpha_j^m \equiv \alpha_j, \;\; for \;\; j = \{1, 2, 3\}$

This assumption is crucial in order to obtain the analytical results derived below. I deviate in this respect from the model derived in Acemoglu and Guerrieri (2008) which relies on differing input shares generating sectoral reallocations. However, as will be seen this assumption serves us well as a baseline case and will help us flesh out the mechanisms that are driving our results. Based on this assumption it is clear that optimal resource allocation will require that the marginal products of capital labor and energy are equalized:

$$
\begin{aligned}
w_m \alpha_1 \left(\frac{Y_t}{Y_{mt}}\right)^{\frac{1}{\epsilon}} \frac{Y_{mt}}{K_{mt}} &= w_a \alpha_1 \left(\frac{Y_t}{Y_{at}}\right)^{\frac{1}{\epsilon}} \frac{Y_{at}}{K_{at}} \\
w_m \alpha_2 \left(\frac{Y_t}{Y_{mt}}\right)^{\frac{1}{\epsilon}} \frac{Y_{mt}}{L_{mt}} &= w_a \alpha_2 \left(\frac{Y_t}{Y_{at}}\right)^{\frac{1}{\epsilon}} \frac{Y_{at}}{L_{at}} \\
w_m \alpha_3 \left(\frac{Y_t}{Y_{mt}}\right)^{\frac{1}{\epsilon}} \frac{Y_{mt}}{E_{mt}} &= w_a \alpha_3 \left(\frac{Y_t}{Y_{at}}\right)^{\frac{1}{\epsilon}} \frac{Y_{at}}{E_{at}}
\end{aligned} \tag{12}
$$

Based on these equations I can solve for the optimal capital, labor and energy shares allocated to each sector. This is allocation is given by the following proposition.

Proposition 1 *Assuming equal sectoral factor shares and constant returns to scale the intratemporal factor allocation is determined by*

$$
\Psi(S_t) = \left(\frac{w_a}{w_m}\right)^{\epsilon} \left(\frac{\Omega_a(S_t)}{\Omega^m(S_t)} \frac{A_{at}}{A_{mt}}\right)^{\epsilon - 1}
$$

where $\Psi(S_t) \equiv \frac{K_{at}}{K_{mt}} = \frac{L_{at}}{L_{mt}} = \frac{E_{at}}{E_{mt}}$ □

Proof see appendix ■

The following corollary also follows immediately from the above proposition

Corollary 1 *If* $w_a \Omega_a A_a < w_m \Omega_m A_m$ *at some point in time a higher value for the elasticity of substitution* ϵ *would allocate more resources to the manufacturing sector (*$\Psi(S_t)$ *decreases) and vice versa.* □

Proof Applying the envelope theorem I have

$$
\frac{d\Psi(S_t)}{d\epsilon} = \left(\frac{w_a}{w_m}\right)^{\epsilon} \left(\frac{\Omega_a A_a}{\Omega_m A_m}\right)^{\epsilon - 1} \ln\left(\frac{w_a}{w_m} \frac{\Omega_a A_a}{\Omega_m A_m}\right) \tag{14}
$$

which is negative *iff* $w_a \Omega_a A_a < w_m \Omega_m A_m$. ■

The above proposition and the following corollary gives us an important heads up regarding how resources will be allocated within the two sectors. Depending on the size of ϵ the sectoral ratio of relative damages to total factor productivity multiplied by the distribution parameter will determine the direction of resource flow. Consider first the case when $\epsilon < 1$ so that the two intermediate goods are complements in production. Then the agricultural sector will be relatively larger if and only if $w_a \Omega_a A_a < w_m \Omega_m A_m$ and smaller if and only if $w_a \Omega_a A_a > w_m \Omega_m A_m$. Further, the total productivity of the two sectors will be determined by the terms $\Omega_a A_a$ and $\Omega_m A_m$, which are both endogenous and time dependent, implying that they will determine which of these two sectors is expanding and which is contracting with time.[7] Making use of proposition 1 the final goods production can now be substantially simplified. Together with the market clearing conditions I can now write the final goods production function as

$$\tilde{Y}_t = \Gamma_t(S_t) K_t^{\alpha_1} L_t^{\alpha_2} E_t^{\alpha_3} \tag{15}$$

where

$$\Gamma_t(S_t) = \left(w_m \Gamma_{mt}^{(\epsilon-1)/\epsilon} + w_a \Gamma_{at}^{(\epsilon-1)/\epsilon} \right)^{\epsilon/(\epsilon-1)} \tag{16}$$

and

$$\Gamma_{at} = \Omega_a(S_t) A_{at} \left(\frac{\Psi(S_t)}{1+\Psi(S_t)} \right) \tag{17}$$

$$\Gamma_{mt} = \Omega^m(S_t) A_{mt} \left(\frac{1}{1+\Psi(S_t)} \right) \tag{18}$$

As can be seen from the above equations the solution to the static equilibrium will simplify the dynamic analysis greatly since final goods production is now a function of only carbondioxide S and the aggregate resource inputs $\{K, L, E\}$.

2.3 Dynamic equilibrium

In order to proceed with analytical results also in derivation of the dynamic equilibrium I will also assume that utility is logarithmic and a capital depreciation rate of a hundred percent.

Assumption 2 $U(C_t) = \ln(C_t),\ \delta = 1$

Logarithmic preferences is rather standard and a common assumption in many models. For example, the earlier climate economy models developed by William Nordhaus all featured logarithmic preferences (see e.g. Nordhaus (1994); Nordhaus and Boyer (2000)). As the main purpose of this paper is more qualitative than quantitative in nature I will not spend time on discussing the robustness of the results to this assumption. However, judging from the results derived in this paper and based on the work of Acemoglu and Guerrieri (2008)[8], modifying this assumption should not affect the qualitative results derived here regarding structural transformation. Golosov *et al.* (2011) also use and discuss these assumptions. They argue that in particular for longer time periods (10 years) suggests a lower curvature of the utility function.

A depreciation rate of a hundred percent is large even for a ten year period. Golosov *et al.* (2011) claim that this does not affect the results of their model remarkably. Together these assumptions are convenient in these types models since it is well known that as long as aggregate capital can be factored out of the production function the saving rate will become a constant.[9]

[7] It is interesting to see that the expression of relative income shares proposition 1 corresponds, with exception of the damage function, precisely to the the ratio of consumption expenditure on a consumption good to consumption expenditure on the manufacturing (capital building) good given by equation (10) in Ngai and Pissarides (2007). The add on here is the climate externality that heterogeneously effects productivity within each sector.

[8] In particular, see proposition 1 and 2 of this paper.

[9] This assumption was first used by Brock and Mirman (1972) to provide a closed-form solution in a stochastic growth setting.

Given this assumption and the results derived from the static equilibrium we can now write down the lagrangian of the dynamic problem facing the social planner

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t [\ln(\tilde{Y}_t - K_{t+1}) + \lambda_{Rt}(R_t - E_t - R_{t+1}) + \lambda_{St}((1-\varphi)S_t + E_t - S_{t+1})] \tag{19}$$

Taking the F.O.C w.r.t. K_{t+1} we have

$$\mathcal{L}_{K_{t+1}} = -\beta_t \frac{1}{C_t} + \beta_{t+1} \frac{1}{C_{t+1}} \alpha_1 \frac{\tilde{Y}_t}{K_{t+1}} = 0 \tag{20}$$

which gives us

$$\frac{C_{t+1}}{C_t} = \beta \alpha_1 \frac{\tilde{Y}_{t+1}}{K_{t+1}} \tag{21}$$

which gives us the following consumption/investment rule:

$$C_t = (1 - \beta\alpha_1)\tilde{Y}_t \tag{22}$$

$$K_{t+1} = \beta\alpha_1 \tilde{Y}_t \tag{23}$$

From the above calculations we see that optimal investment in capital K remains a fixed fraction $\beta\alpha_1$ of manufacturing production over time. Hence it is unaffected by changes to the parameters in the instantaneous utility function such as the elasticity of substitution ϵ between the two consumption goods.

Concerning optimal fuel use I now proceed with the first order conditions w.r.t. the fossil fuel:

$$\mathcal{L}_{R_{t+1}} = -\beta^t \lambda_{Rt} + \beta^{t+1} \lambda_{Rt+1} = 0 \tag{24}$$

$$\mathcal{L}_{E_t} = \alpha_3 \frac{1}{C_t} \frac{\tilde{Y}_t}{E_t} - \lambda_{Rt} + \lambda_{St} = 0 \tag{25}$$

$$\mathcal{L}_{S_{t+1}} = \beta^{t+1} \frac{1}{C_{t+1}} \frac{\partial \Gamma_{t+1}}{\partial S_{t+1}} K_{t+1}^{\alpha_1} L_{t+1}^{\alpha_2} E_{t+1}^{\alpha_3} - \beta^t \lambda_{St} + \beta^{t+1} \lambda_{St+1}(1-\varphi) = 0 \tag{26}$$

From (26) we have:

$$\lambda_{St} = \beta \frac{1}{C_{t+1}} \frac{\partial \Gamma_{t+1}}{\partial S_{t+1}} \frac{\tilde{Y}_{t+1}}{\Gamma_{t+1}} + \beta \lambda_{St+1}(1-\varphi)$$

and thus

$$\lambda_{St} = \sum_{s=1}^{\infty} (1-\varphi)^{s-1} \beta^s \left(\frac{1}{C_{t+s}} \frac{\partial \Gamma_{t+s}}{\partial S_{t+s}} \frac{\tilde{Y}_{t+s}}{\Gamma_{t+s}} \right) + \lim_{s\to\infty} (1-\varphi)^{s-1} \beta^s \left(\frac{1}{C_{t+s}} \frac{\partial \Gamma_{t+s}}{\partial S_{t+s}} \frac{\tilde{Y}_{t+s}}{\Gamma_{t+s}} \right) \tag{27}$$

By the transversality condition the limiting term is zero and if we express the marginal damage cost λ_{St} in terms of present day consumption $\Lambda_{St} \equiv \lambda_{St}/U'(C_t)$ we get an expression similar to equation (12) of Golosov *et al.* (2011).

$$\Lambda_{St} = \sum_{s=1}^{\infty} (1-\varphi)^{s-1} \beta^s \left(C_t \frac{\tilde{Y}_{t+s}}{C_{t+s}} \frac{\partial \Gamma_{t+s}}{\partial S_{t+s}} \frac{1}{\Gamma_{t+s}} \right) \tag{28}$$

This formula is more complex than the one derived in their paper. In particular the formula does not depend on merly the saving rate but also on fossil fuel use thru the term $\frac{\partial \Gamma_{t+s}}{\partial S_{t+s}} \frac{1}{\Gamma_{t+s}}$

which I will refer to as the "relative damage term". However, as will be shown in the next section this term is not a complete black box and we can learn alot about its properties by analyzing how it is affected by changes in its parameters. Further, if we express the marginal externality costs of emissions as a proportion of GDP i.e. $\hat{\Lambda}_{St} \equiv \Lambda_{St}/\tilde{Y}_t$ and make use of the consumption rule we get a simpler expression which is independent of saving and well suited for examining the role of the elasticity of substitution for climate damages.

$$\hat{\Lambda}_{St} = \sum_{s=1}^{\infty}(1-\varphi)^{s-1}\beta^s\left(\frac{\partial\Gamma_{t+s}}{\partial S_{t+s}}\frac{1}{\Gamma_{t+s}}\right) \tag{29}$$

2.3.1 The marginal externality cost of CO_2 and the elasticity of substitution

From (2.2.1) we saw how allocation of factor inputs depended on technical and climatic change within the two sectors and the role played by the elasticity of substitution. In expression (29) we see that marginal climate damages also depends on the elastictity of substitution through the relative damage term $\frac{\partial\Gamma_{t+s}}{\partial S_{t+s}}\frac{1}{\Gamma_{t+s}}$. Hence the marginal externality cost of atmospheric carbon dioxide will depend upon both the substitution possibilities amongst the two sectors and how damages are spread between them. The following proposition is usefull in order to understand how this works.

Proposition 2 *For* $0 \leq \epsilon < \infty$ *the marginal externality cost of carbon dioxide per unit of GDP* (29) *is always negative and bounded above and below by the marginal damages within each sector.*

$$\hat{\Lambda}_{S_t}(\epsilon) \in \left[\hat{\Lambda}_{S_t}(0), \hat{\Lambda}_{S_t}(\infty)\right] \tag{30}$$

where

$$\hat{\Lambda}_{S_t}(0) = \sum_{s=1}^{\infty}(1-\varphi)^{s-1}\beta^s\frac{(\Omega_a A_{at})^{-1}\frac{\Omega_a'}{\Omega_a} + \Omega_m A_{mt})^{-1}\frac{\Omega_m'}{\Omega_m}}{(\Omega^m A_{mt})^{-1} + (\Omega_a A_{at})^{-1}} \tag{31}$$

$$\hat{\Lambda}_{S_t}(\infty) = \begin{cases} \sum_{s=1}^{\infty}(1-\varphi)^{s-1}\beta^s\frac{\Omega_a'}{\Omega_\varphi} & \text{if } w_a\Omega_a A_a > w_m\Omega_m A_m \\ \sum_{s=1}^{\infty}(1-\varphi)^{s-1}\beta^s\frac{\Omega_m}{\Omega_m} & \text{if } w_a\Omega_a A_a < w_m\Omega_m A_m \end{cases} \tag{32}$$

there are four cases to consider:

$$\text{i) } w_a\Omega_a A_a > w_m\Omega_m A_m \text{ and } \frac{\Omega_a'}{\Omega_a} < \frac{\Omega_m'}{\Omega_m} \text{ then } \hat{\Lambda}_{S_t}(\infty) < \hat{\Lambda}_{S_t}(0) : \hat{\Lambda}_{S_t} \text{ decreasing in } \epsilon$$

$$\text{ii) } w_a\Omega_a A_a > w_m\Omega_m A_m \text{ and } \frac{\Omega_a'}{\Omega_a} > \frac{\Omega_m'}{\Omega_m} \text{ then } \hat{\Lambda}_{S_t}(\infty) > \hat{\Lambda}_{S_t}(0) : \hat{\Lambda}_{S_t} \text{ increasing in } \epsilon$$

$$\text{iii) } w_a\Omega_a A_a < w_m\Omega_m A_m \text{ and } \frac{\Omega_a'}{\Omega_a} < \frac{\Omega_m'}{\Omega_m} \text{ then } \hat{\Lambda}_{S_t}(\infty) > \hat{\Lambda}_{S_t}(0) : \hat{\Lambda}_{S_t} \text{ increasing in } \epsilon$$

$$\text{iv) } w_a\Omega_a A_a < w_m\Omega_m A_m \text{ and } \frac{\Omega_a'}{\Omega_a} > \frac{\Omega_m'}{\Omega_m} \text{ then } \hat{\Lambda}_{S_t}(\infty) < \hat{\Lambda}_{S_t}(0) : \hat{\Lambda}_{S_t} \text{ decreasing in } \epsilon$$

□

PROOF The proof is derived by examining the limits of $\frac{\partial \Gamma_t}{\partial S_t}\frac{1}{\Gamma_t}$ as $\epsilon \to 0$ and $\epsilon \to \infty$. First, from (17), (18) and proposition (1) we have

$$\Gamma_{at} = \frac{w_a^\epsilon (\Omega_a A_{at})^\epsilon}{w_m^\epsilon \left(\Omega^m A_{mt}\right)^{\epsilon-1} + w_a^\epsilon (\Omega_a A_{at})^{\epsilon-1}}$$
$$\Gamma_{mt} = \frac{w_m^\epsilon (\Omega_m A_{mt})^\epsilon}{w_m^\epsilon \left(\Omega^m A_{mt}\right)^{\epsilon-1} + w_a^\epsilon (\Omega_a A_{at})^{\epsilon-1}}$$

further we can write

$$\frac{\partial \Gamma_t}{\partial S_t}\frac{1}{\Gamma_t} = \gamma_{mt}\frac{\Gamma'_{mt}}{\Gamma_{mt}} + \gamma_{at}\frac{\Gamma'_{at}}{\Gamma_{at}} \tag{33}$$

where

$$\gamma_m \equiv \frac{\partial \Gamma_t}{\partial \Gamma_{mt}}\frac{\Gamma_{mt}}{\Gamma_t} = \frac{w_m \Gamma_m^{\frac{\epsilon-1}{\epsilon}}}{\left(w_m \Gamma_m^{\frac{\epsilon-1}{\epsilon}} + w_a \Gamma_a^{\frac{\epsilon-1}{\epsilon}}\right)} = \frac{w_m^\epsilon (\Omega_m A_{mt})^{\epsilon-1}}{w_m^\epsilon \left(\Omega^m A_{mt}\right)^{\epsilon-1} + w_a^\epsilon (\Omega_a A_{at})^{\epsilon-1}} \tag{34}$$

$$\gamma_a \equiv \frac{\partial \Gamma_t}{\partial \Gamma_{at}}\frac{\Gamma_{at}}{\Gamma_t} = \frac{w_a \Gamma_a^{\frac{\epsilon-1}{\epsilon}}}{\left(w_m \Gamma_m^{\frac{\epsilon-1}{\epsilon}} + w_a \Gamma_a^{\frac{\epsilon-1}{\epsilon}}\right)} = \frac{w_a^\epsilon (\Omega_a A_{at})^{\epsilon-1}}{w_m^\epsilon \left(\Omega^m A_{mt}\right)^{\epsilon-1} + w_a^\epsilon (\Omega_a A_{at})^{\epsilon-1}} \tag{35}$$

further we can also derive

$$\frac{\Gamma'_{at}}{\Gamma_{at}} = \frac{\Gamma_{at}}{\Omega_a A_a}\frac{\Omega'_a}{\Omega_a} + \frac{\Gamma_{mt}}{\Omega_m A_m}\left(\epsilon\frac{\Omega'_a}{\Omega_a} - (\epsilon-1)\frac{\Omega'_m}{\Omega_m}\right) = \gamma_{at}\frac{\Omega'_a}{\Omega_a} + \gamma_{mt}\left(\epsilon\frac{\Omega'_a}{\Omega_a} - (\epsilon-1)\frac{\Omega'_m}{\Omega_m}\right) \tag{36}$$

$$\frac{\Gamma'_{mt}}{\Gamma_{mt}} = \frac{\Gamma_{mt}}{\Omega_m A_m}\frac{\Omega'_m}{\Omega_m} + \frac{\Gamma_{at}}{\Omega_a A_a}\left(\epsilon\frac{\Omega'_m}{\Omega_m} - (\epsilon-1)\frac{\Omega'_a}{\Omega_a}\right) = \gamma_{mt}\frac{\Omega'_m}{\Omega_m} + \gamma_{at}\left(\epsilon\frac{\Omega'_m}{\Omega_m} - (\epsilon-1)\frac{\Omega'_a}{\Omega_a}\right) \tag{37}$$

substituting (36) and (37) into (33) we have

$$\begin{aligned}
\frac{\partial \Gamma_t}{\partial S_t}\frac{1}{\Gamma_t} &= \gamma_{mt}\left(\gamma_{mt}\frac{\Omega'_m}{\Omega_m} + \gamma_{at}\left(\epsilon\frac{\Omega'_m}{\Omega_m} - (\epsilon-1)\frac{\Omega'_a}{\Omega_a}\right)\right) + \gamma_{at}\left(\gamma_{at}\frac{\Omega'_a}{\Omega_a} + \gamma_{mt}\left(\epsilon\frac{\Omega'_a}{\Omega_a} - (\epsilon-1)\frac{\Omega'_m}{\Omega_m}\right)\right) \\
&= \gamma_{mt}^2\frac{\Omega'_m}{\Omega_m} + \gamma_{at}^2\frac{\Omega'_a}{\Omega_a} + \gamma_{mt}\gamma_{at}\left(\frac{\Omega'_a}{\Omega_a} + \frac{\Omega'_m}{\Omega_m}\right) = \gamma_{at}\left(\gamma_{at}+\gamma_{mt}\right)\frac{\Omega'_a}{\Omega_a} + \gamma_{mt}\left(\gamma_{at}+\gamma_{mt}\right)\frac{\Omega'_m}{\Omega_m} \\
&= \gamma_{at}\frac{\Omega'_a}{\Omega_a} + \gamma_{mt}\frac{\Omega'_m}{\Omega_m}
\end{aligned}$$

from our assumptions on Ω_a and Ω_m we see that $\frac{\partial \Gamma_t}{\partial S_t}\frac{1}{\Gamma_t}$ is always negative implying that marginal damages (29) are also negative.

The limit as $\epsilon \to 0$:

Using standard limit rules for products we evaluate the γ_{at} and γ_{mt} of (33) seperatly.

$$\lim_{\epsilon\to 0}\gamma_{at} = \frac{(\Omega_a A_{at})^{-1}}{\left(\Omega^m A_{mt}\right)^{-1} + (\Omega_a A_{at})^{-1}}$$
$$\lim_{\epsilon\to 0}\gamma_{mt} = \frac{(\Omega_m A_{mt})^{-1}}{\left(\Omega^m A_{mt}\right)^{-1} + (\Omega_a A_{at})^{-1}}$$

this implies that

$$\lim_{\epsilon\to 0}\frac{\partial\Gamma_t}{\partial S_t}\frac{1}{\Gamma_t}=\frac{(\Omega_a A_{at})^{-1}\frac{\Omega_a'}{\Omega_a}+\Omega_m A_{mt})^{-1}\frac{\Omega_m'}{\Omega_m}}{(\Omega^m A_{mt})^{-1}+(\Omega_a A_{at})^{-1}} \tag{38}$$

The limit as $\epsilon \to \infty$:
From (34) and (35) the limits of γ_a and γ_m follow directly:

$$\lim_{\epsilon\to\infty}\gamma_m=\lim_{\epsilon\to\infty}\frac{1}{1+\frac{\Omega_m A_m}{\Omega_a A_a}\left(\frac{w_a\Omega_a A_a}{w_m\Omega_m A_m}\right)^{\epsilon}}=\begin{cases}0 & \text{if } w_a\Omega_a A_a > w_m\Omega_m A_m\\ 1 & \text{if } w_a\Omega_a A_a < w_m\Omega_m A_m\end{cases} \tag{39}$$

$$\lim_{\epsilon\to\infty}\gamma_a=\lim_{\epsilon\to\infty}\frac{1}{1+\frac{\Omega_a A_a}{\Omega_m A_m}\left(\frac{w_m\Omega_m A_m}{w_a\Omega_a A_a}\right)^{\epsilon}}=\begin{cases}1 & \text{if } w_a\Omega_a A_a > w_m\Omega_m A_m\\ 0 & \text{if } w_a\Omega_a A_a < w_m\Omega_m A_m\end{cases} \tag{40}$$

hence we can conclude that

$$\lim_{\epsilon\to\infty}\frac{\partial\Gamma_t}{\partial S_t}\frac{1}{\Gamma_t}=\begin{cases}\frac{\Omega_a'}{\Omega_a} & \text{if } w_a\Omega_a A_a > w_m\Omega_m A_m\\ \frac{\Omega_m'}{\Omega_m} & \text{if } w_a\Omega_a A_a < w_m\Omega_m A_m\end{cases} \tag{41}$$

Finally, to prove that these limits of ϵ also bind the marginal damages (29) between their limiting values we make use of the implicit function theorem and take the derivative w.r.t. ϵ.

$$\frac{d\frac{\partial\Gamma_t}{\partial S_t}\frac{1}{\Gamma_t}}{d\epsilon}=\frac{w_m{}^e(\Omega_m A_m)^{e+1}w_a{}^e(\Omega_a A_a)^{e+1}}{\left(w_m{}^e(\Omega_m A_m)^e(\Omega_a A_a)+w_a{}^e(\Omega_a A_a)^e(\Omega_m A_m)\right)^2}\ln\left(\frac{w_a\Omega_a A_a}{w_m\Omega_m A_m}\right)\left(\frac{\Omega_a'}{\Omega_a}-\frac{\Omega_m'}{\Omega_m}\right)$$

From this comparative static it is straightforward to see that $\frac{\partial\Gamma_t}{\partial S_t}\frac{1}{\Gamma_t}$ is monotonically increasing or decreasing in epsilon depending on the sign of both $\left(\frac{\Omega_a'}{\Omega_a}-\frac{\Omega_m'}{\Omega_m}\right)$ and $\ln\left(\frac{w_a\Omega_a A_a}{w_m\Omega_m A_m}\right)$ which gives the four case i-iv. ■

The main intuition behind Proposition (2) can be attained by examining figure (1). The figure plots the marginal damage term $\frac{\partial\Gamma_t}{\partial S_t}\frac{1}{\Gamma_t}$ for different values of ϵ and w_m. Each line going from the flat horizontal line in the middle of the figure to the more s-shaped line going from -2×10^{-3} to -1×10^{-3} has a different value for the elasticity of substitution ϵ. Here the horizontal line has $\epsilon = 0$ while the most s-shaped line has an $\epsilon = 4$. The figure was plotted assuming a constant $A_{mt} = A_{at} = 1$ and $S_t = 583$. The damage functions were given an exponential form $\Omega_m = e^{-\theta_m S_t}$ and $\Omega_a = e^{-\theta_a S_t}$ with $\theta_m = 0.001$ and $\theta_a = 0.002$. I am thus assuming that a temperature increase will have a larger impact on the agricultural sector.

The figure confirms the results of Proposition (2). For an elasticity of substitution equal to zero the marginal damage term is merely a weighted average of the marginal damages within each sector which is independent of the distribution parameter. However, as ϵ increases the marginal damage term moves towards the boundaries discussed in Proposition (2) where the direction depends on the conditions $i - iv$. Numerical examination of the behavior when $\epsilon \to \infty$ shows that the steepness of the s-shape increases with an increasing ϵ with a threshold located about the dashed vertical line.

Concerning the interpretation, there are two cases to consider. First, consider the case where w_m lies in the region to the right of the dashed vertical line in figure 1. In this

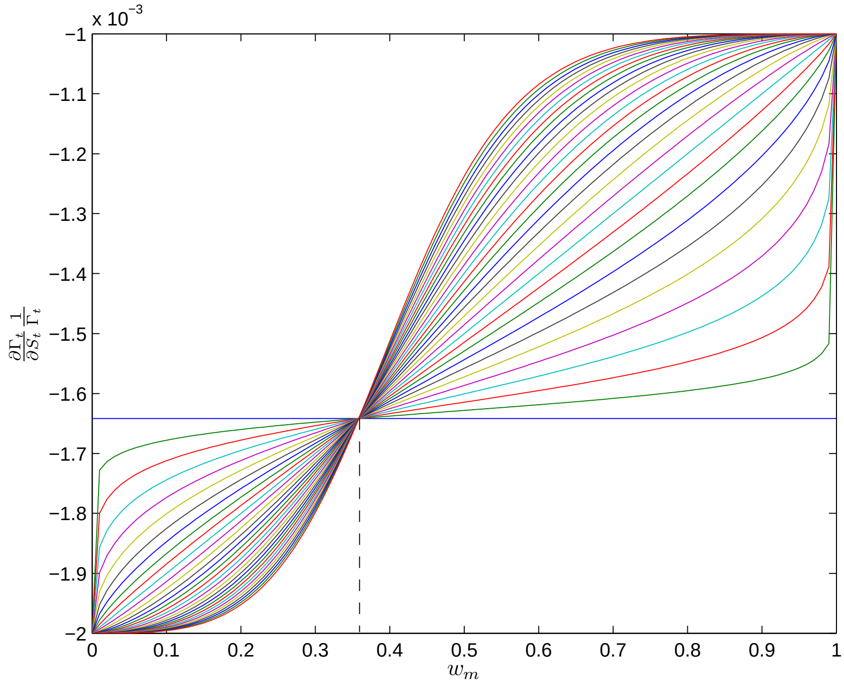

Figure 1: Behaviour of the relative damage term $(\frac{\partial \Gamma_t}{\partial S_t}\frac{1}{\Gamma_t})$ for different values of ϵ and w_m. Flatter lines depict small values of ϵ while increasing s-shaped correspond to higher values of ϵ. The figure was plotted assuming a constant $A_{mt} = A_{at} = 1$ and $S_t = 583$. The damage functions were given an exponential form $\Omega_m = e^{-\theta_m S_t}$ and $\Omega_a = e^{-\theta_a S_t}$ with $\theta_m = 0.001$ and $\theta_a = 0.002$.

case we can see from the figure that for each fixed value of w_m the relative damage term increases with higher values of ϵ. Hence, judging from proposition (2) we can conclude that this should correspond to the case where $w_a \Omega_a A_a < w_m \Omega_m A_m$. Further, by proposition (1) we also see that in this case resources are moving in the direction of the manufacturing sector and since a large value of ϵ corresponds to a high degree of substitutability this simply means that we are allocating more resources to the sector which suffers the least by climate change.

The second case corresponds to small values of w_m lying to the right of the dashed line. Here, the small value of w_m gives little weight to the manufacturing sector. This implies that when substitutability increases ($\epsilon \uparrow$) the majority of resources are allocated to the agricultural sector since it despite damages makes a greater contribution to overall production. However, because damages are larger in this sector this implies that the marginal damage term $\frac{\partial \Gamma_t}{\partial S_t}\frac{1}{\Gamma_t}$ also becomes larger. Hence, similar to the results obtained by Ngai and Pissarides (2007), it is not only the size of the elasticity of substitution that matters but also the size of the distribution parameter of the CES function that determines the direction of resource allocation and hence as shown here, the externality cost of carbon dioxide.

Finally, it is interesting to see what happens in the cobb-douglas case when $\epsilon \to 1$. In this case it is straightforward to see that $\gamma_m = w_m$ and $\gamma_a = w_a$ implying that the limit follows directly

$$\lim_{\epsilon \to 1} \frac{\partial \Gamma_t}{\partial S_t}\frac{1}{\Gamma_t} = w_a \frac{\Omega_a'}{\Omega_a} + w_m \frac{\Omega_m'}{\Omega_m}$$

Hence we see that the marginal damage term becomes a linear function of the distribution parameters implying that if a sector is valued more in final goods production a higher vulnerability to climate change in this sector also implies an overall higher marginal damage.

2.3.2 Optimal fossil-fuel use

Next we turn to the optimality conditions (24) and (25) which together become

$$\beta^t \left(\alpha_3 \frac{1}{C_t} \frac{\tilde{Y}_t}{E_t} + \lambda_{St} \right) = \beta^{t+1} \left(\alpha_3 \frac{1}{C_{t+1}} \frac{\tilde{Y}_{t+1}}{E_{t+1}} + \lambda_{St+1} \right) \tag{42}$$

Rewriting this expression and making use of (21) and (28) we have

$$\alpha_1 \frac{\tilde{Y}_{t+1}}{K_{t+1}} = \frac{\alpha_3 \frac{\tilde{Y}_{t+1}}{E_{t+1}} + \Lambda_{St+1}}{\alpha_3 \frac{\tilde{Y}_t}{E_t} + \Lambda_{St}} \tag{43}$$

This is a variant of the Hotelling rule stating that the return on capital should be set equal to the return of postponing the extraction. The difference between this expression and the original rule is the appearence Λ_S in the numerator and denominator of the right hand side. Recall that Λ_S is negative hence if Λ_S is falling due to accummulating CO_2 so that $\Lambda_{St+1} < \Lambda_{St}$ then this implies that returns to investment must also be falling. In the market version the price of fossil fuel must equal its marginal product in equilibrium. Hence, in the variant above the terms $\alpha_3 \frac{\tilde{Y}_t}{E_t} + \Lambda_{St}$ can be interpreted as an externality adjusted fossil fuel price at time t.

Further by the investment and consumption rule (22) and (23) we can write

$$\frac{1}{\beta} = \frac{\alpha_3 \frac{1}{E_{t+1}} + \hat{\Lambda}_{St+1}}{\alpha_3 \frac{1}{E_t} + \hat{\Lambda}_{St}} \tag{44}$$

We can now see how a constant saving rate greatly simplifies the analysis. As can be seen from (44) the optimal path of fossil fuel use can now be fully determined given the appropriate end constraints and that carbon dioxide accumulation depends on past values of emissions according to (9). Hence, the decision regarding capital accumulation has been separated out from that evolving energy use.

2.4 The decentralized economy

I will now characterize the how the optimum can be implemented using either a *ad-valorem* (τ_t) or a *per-unit taxes* (θ_t). I assume that the government taxes resources firms in order to implement the optimum. Consider first the problem of the the representive household/individual problem

$$\max \sum_{t=0}^{\infty} \beta^t U(C_t),$$
$$s.t.\ \ C_t + K_{t+1} = r_t K_t + w_t + \Pi_t^e + G_t \tag{45}$$

as usual households are assumed to own both production and resource extraction firms and hence profit by renting out capital at the rate r_t, labor at the wage rate w_t, profits from resource extraction $\Pi_t^e \equiv (p_{E_t} - \theta_t)(1-\tau_t)E_t$ and government transfers $G_t \equiv \tau_t p_{E_t} E_t + \theta_t E_t$ where p_{E_t} denotes the before tax price on fossil fuel. The f.o.c. for K_{t+1} is

$$\frac{C_{t+1}}{C_t} = \beta r_{t+1} \tag{46}$$

Second, the representive firm within the each sector faces the following problem

$$\max_{K_{it}, L_{it}, E_{it}} p_{y_{it}} Y_{it} - r_t K_{it} - w_t L_{it} - p_{E_t} E_{it}, \quad \forall i \in \{m, a\}$$

with the f.o.c. given by

$$r_t = p_{y_{it}} \frac{\partial Y_t}{\partial K_{it}}, \quad w_t = p_{y_{it}} \frac{\partial Y_t}{\partial L_{it}}, \quad p_{E_t} = p_{y_{it}} \frac{\partial Y_t}{\partial E_{mt}} \tag{47}$$

Final good production is also done under profit maximization and perfect competition implying that the marginal product of each good will equal its price. Normalizing the price of the final good to one we thus have that:

$$p_{y_{it}} = w_i \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\epsilon}} \tag{48}$$

inserting this into (47) we see that

$$\begin{aligned}
r_t &= w_m \left(\frac{Y_t}{Y_{mt}} \right)^{\frac{1}{\epsilon}} \frac{\partial Y_t}{\partial K_{mt}} = w_a \left(\frac{Y_t}{Y_{at}} \right)^{\frac{1}{\epsilon}} \frac{\partial Y_t}{\partial K_{at}} \\
w_t &= w_m \left(\frac{Y_t}{Y_{mt}} \right)^{\frac{1}{\epsilon}} \frac{\partial Y_t}{\partial L_{mt}} = w_a \left(\frac{Y_t}{Y_{at}} \right)^{\frac{1}{\epsilon}} \frac{\partial Y_t}{\partial L_{at}} \\
p_{E_t} &= w_m \left(\frac{Y_t}{Y_{mt}} \right)^{\frac{1}{\epsilon}} \frac{\partial Y_t}{\partial E_{mt}} = w_a \left(\frac{Y_t}{Y_{at}} \right)^{\frac{1}{\epsilon}} \frac{\partial Y_t}{\partial E_{at}}
\end{aligned}$$

These profit maximizing conditions are the same as in the planner solution with the solution derived in appendix and stated explicitly in (15). From this solution we thus know that:

$$r_t = \frac{\partial \tilde{Y}_t}{\partial K_t} = \frac{\partial \tilde{Y}_t}{\partial K_{mt}} = \alpha_1 \Gamma_t(S_t) K_t^{\alpha_1 - 1} L_t^{\alpha_2} E_t^{\alpha_3} \tag{49}$$

which implies that the same consumption/investment rules will apply as in the social planner solution i.e. (22) and (23) hold also in the decentralized solution.

Third, the representive resource extraction firm solves the problem

$$\begin{aligned}
&\max_{R_{t+1}} \sum_{t=0}^{\infty} (p_{E_t} - \theta_t)(1 - \tau_t) E_t \left(\prod_{s=0}^{t} r_s \right)^{-1} \\
&s.t. \; R_{t+1} - R_t = -E_t, \quad R_{t+1} \geq 0
\end{aligned}$$

Once again, this gives us a hotelling type of formula

$$r_{t+1} = \frac{(p_{E_{t+1}} - \theta_{t+1})(1 - \tau_{t+1})}{(p_{E_t} - \theta_t)(1 - \tau_t)} \tag{50}$$

I have now characterized the decentralized solution. Next, in order to implement the planning solution we have to set the private return of not using fossil fuels equal to its social return.

Proposition 3 *The optimal tax can be implemented by either setting*

$$\theta_t = -\Lambda_{st} \;\; \textit{and} \; \tau_t = \tau \;\; \forall t$$

or by setting

$$\tau_t = -\frac{\Lambda_{st}}{\partial \tilde{Y}_t / \partial E_t} \;\; \textit{and} \; \theta_t = 0 \qquad \square$$

PROOF Setting the rental price of capital from (50) equal to that of the marginal product of capital in from the planner problem (43) we can write

$$\frac{(p_{E_{t+1}} - \theta_{t+1})(1 - \tau_{t+1})}{(p_{E_t} - \theta_t)(1 - \tau_t)} = \frac{\alpha_3 \frac{\tilde{Y}_{t+1}}{E_{t+1}} + \Lambda_{St+1}}{\alpha_3 \frac{\tilde{Y}_t}{E_t} + \Lambda_{St}}$$

from this expression its immediate that if $\tau_t = \tau$ then $\theta_t = \Lambda_{st}$ implements the planner optimum. Likewise, if $\theta_t = 0$ then $\tau_t = \frac{\Lambda_{st}}{\partial \tilde{Y}_t / \partial E_t}$ implements the optimum. ■

3 Numerical analysis

In this section I undertake a simple calibration and simulation exercise in order to illustrate how the optimal fossil fuel consumption problem is affected by underlying model assumptions. I have chosen to calibrate two sets of parameter estimates based on data from India and the U.S. economy respectively. These countries differ vastly in the size of their agricultural sector compared to other sectors of the economy and thus serves well as illustrative examples of economies having different sectoral compositions.[10] The result of this calibration excercise can thus be seen as the policy suggestion resulting from two types of planners trying to adress the economic realities adherent in two economies under different levels of economic development.[11]

3.1 Model calibration

In the analytical section above I neglected that economic damages from climate change are typically measured as a function of temperature. This was done in order to avoid complexity that does not impact on qualitative modeling results. A simple way to correct for this simplification is thru the Arrhenius equation, after the Swedish physicist Svante Arrhenius (1859-1927), which states that when CO_2 increases in a geometric progression, the augmentation of the temperature will increase in a nearly arithmetic progression (see e.g. IPCC (2001)). This fairly simple way of capturing the relationship between CO_2 is still in use today in simplified climate system representations approximating global temperature rise from a doubling of atmospheric CO_2 (see e.g. Nordhaus (2007))

$$T_t \equiv T(S_t) = \lambda \ln \left(1 + \frac{S_t}{\bar{S}}\right) / \ln 2$$

[10]For example, the world bank estimates that agricultures share of value added amounts to approximately 1% of the U.S. economy while in India the share is approximately 20% (http://data.worldbank.org/indicator/NV.AGR.TOTL.ZS?page=1).

[11]It should be noted that the assumption of a constant elasticity of substitution is not uncontroversial. Already in the seminal article Arrow *et al.* (1961) it was pointed out that given the systematic intersectoral differences in the elasticity of substitution and in income elasticities of demand, the possibility arises that the process of economic development itself might shift the over-all elasticity of substitution. Hence, estimates for the elasticity of substitution may vary greatly depending on the level of development.

where $\bar{S}$ is the pre-industrial atmospheric CO_2 level and λ is a climate sensitivity parameter. Following Nordhaus (2007) (see page 13) I assume a preindustrial concentration of carbon dioxide of $\bar{S} \approx 596Gtc$ (gigagtons of carbon).[12] Taking the fairly standard value of 3 for λ we have that for a doubling of CO_2 in the atmosphere temperature will rise by approximately 3 degrees Celsius.[13] Tans (2011) reports a current CO_2 concentration as of December 2011 of $831Gtc$. Hence, using the above formula this implies that global temperature has increased by approximately 1.44 degrees since pre-industrial levels lies well in the ballpark of more accurate climate model runs found in e.g. the IPCC (2007). Concerning the carbon dioxide accumulation equation (9) two parameters need to be calibrated. First, the so called airborne fraction ξ which is the fraction of anthropogenic carbon dioxide emissions that remain in the atmosphere. I model this fraction as constant hence assuming that there is no trend in the biosphere's and oceans ability to absorb human induced emissions. Based on a recent study by Knorr (2009) I set the airborne fraction $\xi = 0.43$.[14] Second, the parameter φ captures the rate at which carbon is absorbed by the deep oceans. Archer (2005) claims that "...75% of an excess atmospheric carbon concentration has a mean lifetime of 300 year and the remaining 25% stays there forever". Although my simple carbon cycle is unable to account for the 25% always remaining in the atmosphere by setting $\varphi = 0.05$ this implies that after 300 years (30 periods) approximately 75% of the carbon dioxide has been removed.

Next, I need to specify the damage functions for the two sectors. Although there exists a large and growing literature concerned with sector specific damages at the local scale it is difficult to find similar assessments at the global scale. Reviewing and aggregating such studies is a large undertaking and lies beyond the scope of the current paper.[15] I follow Nordhaus (2007) here and specify a simple quadratic damage function for each of the two sectors and rely upon the aggregation results found in his accompanying notes for my benchmark estimation.[16] On page 24 of his accompanying notes he presents damage estimates in percent of GDP disaggregated into 12 geographical regions and seven damage sectors including an estimate of catastrophic damages. These damage estimates are then aggregated, using output weights based on predicted 2105 year's GDP levels, into a single estimate of GDP loss due to a 2.5 degree warming of approximately 1.77%. The exact size of these weights are however not a supported part of the DICE model.[17] Following, Nordhaus (2007) I proceed by calibrating two different quadratic damage functions for both the U.S. and Indian economy based on the estimates found in his notes. For the U.S. economy he estimates an economic impact of 0.03% of GDP from the agricultural sector and 0.88% of GDP from the rest of the economy from a 2.5 degree warming. Similarly for the Indian economy he estimates an economic impact of 0.32% of GDP from the agricultural sector and 2.75% of GDP from the rest of the economy. Based on these estimates the damage functions

$$\Omega_a(T_t) = \frac{1}{1+\theta_a T_t^2}, \quad \Omega_m(T_t) = \frac{1}{1+\theta_m T_t^2} \tag{51}$$

are calibrated for the U.S. and Indian economy where the parameters are calculated as

[12] I report atmospheric carbon dioxide in gigatons of carbon (GtC) with a conversion factor 1 ppm by volume of atmospheric CO_2 = 2.13 GtC.

[13] There is considerable uncertainty surrounding the appropriate estimate for climate sensitivity see for example Roe and Baker (2007).

[14] Although there exists several studies have reported an apparent increasing trend in the airborne fraction the study by Knorr (2009) claim that this trend is statistically insignificant.

[15] See e.g. Tol (2009) for a review of such estimates.

[16] See `http://nordhaus.econ.yale.edu/Accom_Notes_100507.pdf`

[17] Personal communication with William Nordhaus.

$\theta_a = 0.0003/2.5^2 \approx 4.8 \times 10^{-5}$ and $\theta_m = 0.0088/2.5^2 \approx 0.0014$ for the U.S. economy and set to $\theta_a = 0.0032/2.5^2 \approx 0.000512$ and $\theta_m = 0.0275/2.5^2 \approx 0.0044$ for the Indian economy.

For the parameters of the agricultural and manufacturing production functions I follow Golosov *et al.* (2011) and set $\alpha_1 = 0.3$, $\alpha_2 = 0.67$ and $\alpha_3 = 0.03$ for both economies. These estimates are fairly standard for manufacturing or as average estimates for the entire economy. However, if land is considered a capital good, empirical studies point to a much larger estimate for the capital income share in the agricultural sector. Valentinyi and Herrendorf (2008) estimate income shares of capital and labor at the sectoral level for the US economy and find that the capital share of the agricultural sector is approximately 50% larger than the capital share of the aggregate economy. For developing countries the share is somewhat smaller (Irz and Roe, 2005). Concerning the TFP growth rates, Martin and Mitra (2001) estimate that the overall growth rate of TFP in manufacturing varies between 1.13% and 1.86% between 2.34% and 2.91% for agriculture for a sample of 50 countries between the years 1967-92. This is one of the few global studies I could find with the same sectoral disaggregation as I consider here. I thus use the mean of these intervals, i.e. $g_m = 0.015$ and $g_a = 0.026$, as my initial benchmark estimates in both economies. It should however be noted that there is a significant amount of disagreement in the literature whether this relatively high rate of TFP growth in agriculture can continue also in the future. The spread of knowledge on the productive use of pesticides and fertilizers in farming increased output considerably in this sector during the second half of the 20th century but it is unclear whether this level of growth can continue at the same pace in the future due to e.g. biophysical constraints in plant life (Ruttan, 2002).

Concerning the elasticity of substitution and distribution parameters I calibrate these following Acemoglu and Guerrieri (2008). Since, Y_{it} corresponds to the quantity produced in sector $i \in \{m, a\}$ the value of output (nominal value added) produced in sector i follows from $Y_{it}^n \equiv p_{it} Y_{it}$ where p_{it} is given by equation (47). This implies the following way of estimating these parameters using a simple linear regression by taking the log of the ratio of sectoral nominal value added and real value added

$$\ln\left(\frac{Y_{mt}^n}{Y_{at}^n}\right) = \ln\left(\frac{w_m}{w_a}\right) + \frac{\epsilon - 1}{\epsilon} \ln\left(\frac{Y_{mt}}{Y_{at}}\right) \tag{52}$$

I estimate the above equation using data from the Groningen Growth and Development Centre (GGDC) 10-sector database for the years 1950-2005 for the U.S. and Indian economy. The data set contains annual series of value added, real value added, and persons employed for 10 broad sectors of the economy (Timmer and de Vries, 2009). I let the *agriculture, forestry, and fishing* sector of the 10 sector database represent the agricultural sector of my model while the aggregate of the remaining sectors excluding government services serves as a proxy for the manufacturing sector. Using these aggregate estimates of real value added and value added I obtain by linear regression from (52) an estimate of $\epsilon \approx 1.62$ with a standard error of 0.147 for the U.S. economy. For the Indian economy the corresponding estimate is $\epsilon \approx 2.13$ with a standard error of 0.15.[18] I thus choose these values as my benchmark estimates and then calibrate w_m to ensure that (52) holds for the year 2005 which gives me an estimate of $w_m \approx 0.95$ for the U.S. economy and $w_m \approx 0.64$ for the Indian economy. The above estimation process has thus given us a benchmark estimate of the elasticity of substitution and distribution parameters. In the result section we will thus also consider alternative calibrations based with an elasticity of substitution above and below these estimates.

[18] Both estimates are significant on the 1% level.

Next, we also need a to calibrate A_{m0} and A_{a0}. As above I consider 2005 as the initial time 0 and proceed as follows. First, the input factor share is determined by making use of (8) and equation (54) in appendix i.e. we have that $K_{a0} = \frac{\Psi_0}{1+\Psi_0}K_0$, $L_{a0} = \frac{\Psi_0}{1+\Psi_0}L_0$ and $E_{a0} = \frac{\Psi_0}{1+\Psi_0}E_0$ where $\Psi_0 \equiv \frac{w_a}{w_m}\left(\frac{Y_{a0}}{Y_{m0}}\right)^{\frac{\epsilon-1}{\epsilon}}$. Substituting these input factors in the agricultural production function and the corresponding manufacturing inputs into the manufacturing production function we have that $Y_{a0} = \frac{\Psi_0}{1+\Psi_0}\Omega_a(T_0)A_{a0}K_0^{\alpha_1}L_0^{\alpha_2}E_0^{\alpha_3}$ and $Y_{m0} = \frac{1}{1+\Psi_0}\Omega_m(T_0)A_{m0}K_0^{\alpha_1}L_0^{\alpha_2}E_0^{\alpha_3}$. Dividing the left and right hand sides of these production functions we can solve for the ratio

$$\frac{A_{a0}}{A_{m0}} = \frac{w_m}{w_a}\frac{\Omega_m(T_t)}{\Omega_a(T_t)}\left(\frac{Y_{a0}}{Y_{m0}}\right)^{1/\epsilon} \tag{53}$$

Using our estimated values for the elasticity of substitution and distribution parameters together with the 2005 real value added estimates for each of the two sectors as initial values we arrive at an estimate of $\frac{A_{a0}}{A_{m0}} \approx 1.46$ for the U.S. economy and $\frac{A_{a0}}{A_{m0}} \approx 0.94$ for the Indian economy. We can thus without loss of generality normalize and set $A_{m0} = 1$ when calculating optimal emission paths based on equation (44). Finally, fossil fuel use in our model also requires an estimate of the current stock R_0. We use the global reserve estimate of $5000GtC$ from Rogner (1997) which also accounts for technical progress in extraction. This estimate is assumed to consist of both coal and oil resources. It should be noted that the way I model fossil fuel production in this paper is highly simplistic as it ignores important issues such as e.g. extraction and refinement costs which are generally much higher for coal than oil. Finally, the last parameter that needs mentioning is the discount factor β which I set equal to 0.985^{10}, where the tenth power is due to the fact that I am considering each time step to be of ten years length. This follows previous work by Nordhaus (2007) and Golosov *et al.* (2011).

3.2 Results

In section 2, I showed, that finding the optimal fossil fuel path involves solving a fairly simple difference equation given by equation (44) which was shown to be a decision independent of the saving policy.[19] In this section I will present some results for two alternative calibrations based on the U.S. and Indian economy. The results for the first calibration based on the estimates for the U.S. economy are given in figure 2.

The figure depicts optimal transition paths for several of the key variables of the economy for a 500 year time horizon. Each figure contains simulation results for three estimates of $\epsilon = 4$ red dash-dotted line and $\epsilon = 0.4$ green dashed line together with a solid blue lines which depicts the paths for the benchmark calibration.[20] As can be seen the simulation results for are almost inseparable for the paths based on the different values of ϵ. This will be differ when we consider the calibration based on India below. Starting from the upper left graph this depicts the optimal fossil fuel use of the economy. As can be seen the path declines over time and approaches zero as $t \to \infty$. The middle upper graph

[19]The numerical simulations are done in Matlab and the code is readily available from the author upon request. The basic solution strategy for equation (44) is as follows: a) Use the decentralized path for E as an initial guess of a solution until some time T (large enough to approximately exhaust the resource) b) Use this path to derive the S_t from (9) and $\hat{\Lambda}_s$ from (29) c) Using $\hat{\Lambda}_s$ use (44) to generate a new path for $E_t + 1, E_t + 2$, etc... which also satisfies the resource constraint (7). d) Use this path to update S_t from (9) and $\hat{\Lambda}_s$ from (29) e) Now repeat from step c) and continue until convergence.

[20]For $\epsilon = 0.4$ and 4, I also recalibrate equation (52) and equation (53) based on the GGDC output data for 2005.

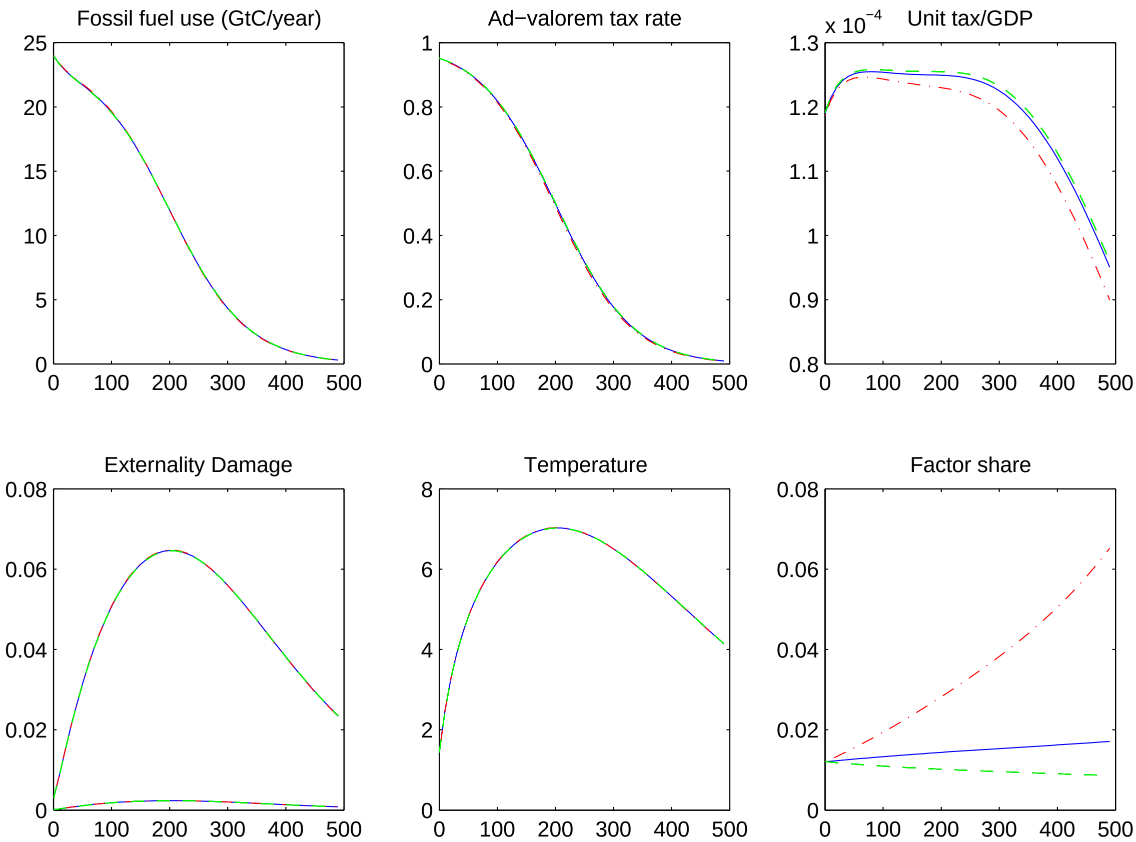

Figure 2: Optimal transition paths for the U.S. economy calibration.

depicts the ad-valorem tax rate. Alternatively we could have referred to this as a unit tax per market price of fossil fuel. Using this terminology taxes thus start at approximately 95% of the market price and remain high for quite some time. The upper left graph depicts the unit tax per unit of GDP. This graph is displayed in a scale that shows off the results for the different values of ϵ. As can be seen the upper line displaying the results for a $\epsilon = 0.4$ implies a slightly higher tax for the entire time period. Likewise, lower line displaying the results for a $\epsilon = 4$ gives the lowest tax. The middle graph (blue line) depicts the benchmark case. Hence, we see that with increasing values of ϵ the optimal tax rate declines. A carefull inspection of the optimal fossil fuel path graph also reveals that larger values of ϵ decreases the optimal fossil fuel use early on. Next the lower left graph displays the externality damage which is the percent of damage to GDP as a function of tempertature increase. The lower line displays the amount of damage coming from the agricultural sector. Damages are lower in this sector due to the fact that it constitutes a smaller proportion of GDP than the manufacturing sector. As can be seen for the manufacturing sector damages peaks at slightly above 6% of GDP after 200 years while the agricultural damages never goes above 1% of GDP. The lower middle graph depicts the global temperature increases that would result following the emissions associated with the optimal path.[21] Finally, the lower rightmost graph depicts the income factor shares corresponding to the expression in proposition 1. As can be seen the differing values for ϵ creates clear trends when it comes to the allocation of factor inputs. Here, the dash-dotted

[21] These temperature levels are by many climate scientists regarded as highly dangerous when it comes to human survival on the planet (see e.g. Hansen (2005); Rockström *et al.* (2010)).

(red line) having an $\epsilon = 4$ creates a clear upward trend in factor input allocation implying that more resources are allocated to agriculture over time. This is due to the larger TFP estimate for the agricultural sector implying that the agricultural sector becomes more productive over time. With an elasticity of substitution larger than one (i.e. substitutes) this implies that factor inputs are flowing in to the more productive sector which can also be seen from the positive slope of the solid blue line. On the contrary the green dashed line is downward sloping implying that resources are flowing into the least productive sector as it continues to grow relatively less productive.

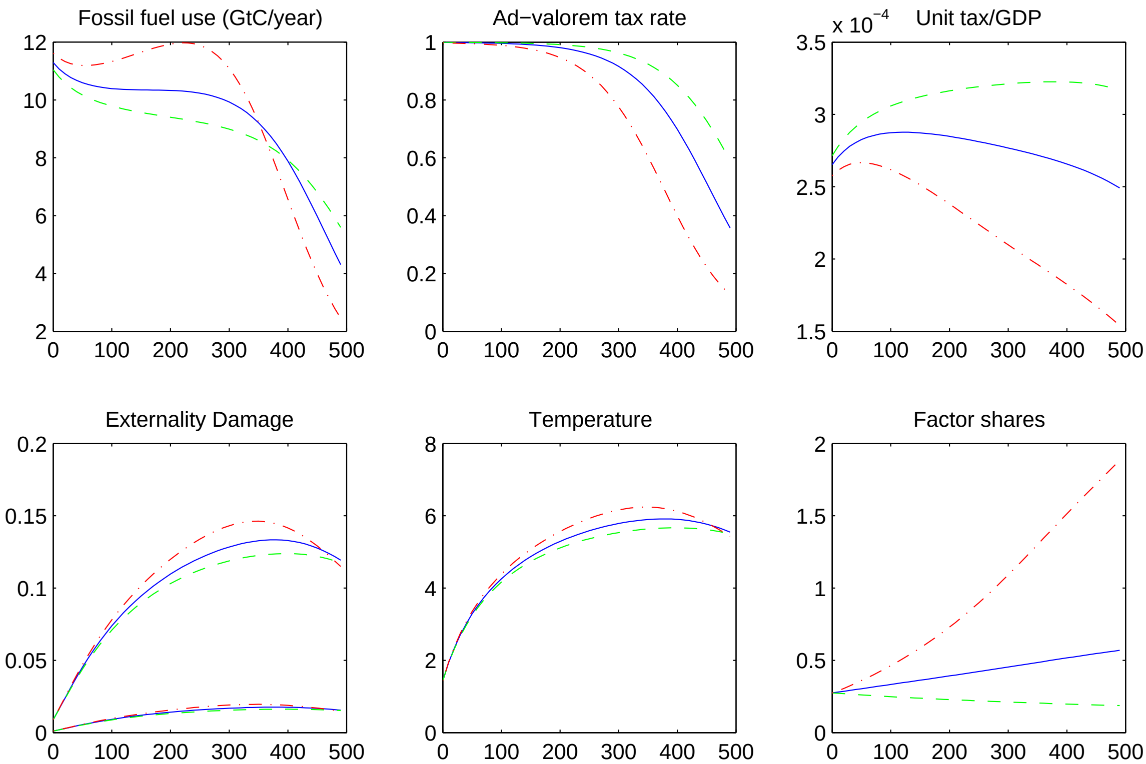

Figure 3: Optimal transition paths for the Indian economy calibration.

Turn now to the Indian economy. Figure 3 displays the corresponding paths of for the variables when the model has been calibrated to match the data from the Indian economy. As can be seen this figure displays a great deal more of variation in all of the depicted variables. One major difference between the Indian and U.S. calibrations was that the damage estimates for a 2.5 degree warming were substantially higher for the Indian economy. A comparison can be made in terms of ad-valorem taxes which for the Indian economy constitutes approximately 100% of the market price at the outset as opposed to 95% for the U.S. economy. This also results in a flatter path of fossil fuel use since the higher damages makes it optimal to postpone fossil fuel consumption to the future. As can be seen from the upper right graph the optimal paths vary a great deal with the size of the elasticity of substitution. The red dash-dotted line with $\epsilon = 4$ the path exhibits a slight u-shape. Here damages early on are significant enough to warrant a lower level of fossil fuel use at the outset than after 200 years. From the lower right graph we see that this is likely to be connected to the increasing factor share implying that resources are flowing into the agricultural sector over time. Since this sector is less plagued with damages this explains why fossil fuel use starts to rise again after approximately 100 years. The final downturn after 250 years is of course due to resource scarcity leading to a rising relative price for fossil fuel consumption. From the lower lefthand graph depicting the externality damage as a percent of GDP we see that damages to the Indian economy are substantially higher than those observed for the U.S. economy peaking at roughly 15% of GDP after 300 years. From the temperature graph we see that this extra damage has however resulted in a more restrictive optimal fossil fuel use policy which keeps the global temperature from rising much higher than 6 degrees as opposed to the near 7 degree peak in the U.S. economy.

4 Concluding remarks

In this paper I have developed a two-sector general equilibrium model featuring a global climate externality arising from the use of fossil fuels in production. I derive analytical and numerical results for the optimal fossil fuel use in the social planner setting and the corresponding unit and ad-valorem tax rates that implement the planner solution in a decentralized market equilibrium. I have shown that when the elasticity of substitution between the two sectors is constant the same economic forces giving rise to structural change also impact the externality costs of climate change. The analytical results reveal the behavior of optimal tax rates subject to sectoral differences in damages, total factor productivity and sectoral weights between the two sectors. It is shown that a higher (lower) elasticity of substitution will result in a higher (lower) optimal unit tax rate if and only if the sectoral weight of the most productive sector, where productive refers to total factor productivity net climate damages, is small (large) enough. The model and results derived here draws upon the results and findings of papers by Ngai and Pissarides (2007), Sterner and Persson (2008), Acemoglu and Guerrieri (2008) and Golosov *et al.* (2011).

Given a set of simplifying assumptions following Golosov *et al.* (2011) I am able to derive i) a simple formula for the marginal externality cost of emissions and ii) a structural change mechanism which is almost identical to demand side mechanism driving the results in Ngai and Pissarides (2007).[22] However, in contrast to Ngai and Pissarides (2007), the mechanism driving structural change in this paper is a combination of damages and productivity assumptions which enter the model as a supply side phenomena.

The numerical section of the paper gives a simulated example based on the U.S. and Indian economy. Calibrating the model to two economies at different stages of development is a crude way of accommodating that systematic inter-sectoral differences in the elasticity of substitution, imply the possibility that the process of economic development itself might shift the over-all elasticity of substitution (Arrow *et al.*, 1961). The results suggest that the elasticity of substitution plays a large role for optimal fossil fuel consumption in the Indian economy where the agricultural sector constitutes approximately 20% of GDP. On the contrary the U.S. economy where the agricultural sector constitutes only 1% of GDP this parameter plays only a peripheral role.

This paper has been a first attempt at developing a climate-economy model where it is possible to explore how substitution possibilities among goods might impact growth and marginal externality costs. The model was developed in the tradition of modern macroeconomic growth models with the intention of making it more accessible to empirical studies exploring the role of substitution possibilities for calculating the costs of climate change. A step for future research could be a more rigid calibration of the model where for example the assumption of equal capital shares are also relaxed.

[22] If the climate externality is removed from the model developed here these mechanisms would have been identical.

A Appendix

A.1 Static problem

PROOF of Proposition (1)
Rewrite equation (12) as:

$$\begin{aligned}\frac{K_{at}}{K_{mt}} &= \frac{w_a}{w_m}\left(\frac{Y_{mt}}{Y_{at}}\right)^{\frac{1}{\epsilon}}\frac{Y_{at}}{Y_{mt}} = \left(\frac{\tilde{w}_a}{\tilde{w}_m}\frac{Y_{at}}{Y_{mt}}\right)^{\frac{\epsilon-1}{\epsilon}} \\ \frac{L_{at}}{L_{mt}} &= \frac{w_a}{w_m}\left(\frac{Y_{mt}}{Y_{at}}\right)^{\frac{1}{\epsilon}}\frac{Y_{at}}{Y_{mt}} = \left(\frac{\tilde{w}_a}{\tilde{w}_m}\frac{Y_{at}}{Y_{mt}}\right)^{\frac{\epsilon-1}{\epsilon}} \\ \frac{E_{at}}{E_{mt}} &= \frac{w_a}{w_m}\left(\frac{Y_{mt}}{Y_{at}}\right)^{\frac{1}{\epsilon}}\frac{Y_{at}}{Y_{mt}} = \left(\frac{\tilde{w}_a}{\tilde{w}_m}\frac{Y_{at}}{Y_{mt}}\right)^{\frac{\epsilon-1}{\epsilon}} \end{aligned} \tag{54}$$

where $w_a \equiv \tilde{w}_a^{\frac{\epsilon-1}{\epsilon}}$ and $w_m \equiv \tilde{w}_m^{\frac{\epsilon-1}{\epsilon}}$. Further solving for the respective ratios we have

$$\begin{aligned}\frac{K_{at}}{K_{mt}} &= \left(\left(\frac{\tilde{w}_a}{\tilde{w}_m}\frac{\Omega_a(S_t)}{\Omega_m(S_t)}\frac{A_{at}}{A_{mt}}\frac{L_{at}^{\alpha_2}}{L_{mt}^{\alpha_2}}\frac{E_{at}^{\alpha_3}}{E_{mt}^{\alpha_3}}\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-\alpha_1(\epsilon-1)}} \\ \frac{L_{at}}{L_{mt}} &= \left(\left(\frac{\tilde{w}_a}{\tilde{w}_m}\frac{\Omega_a(S_t)}{\Omega_m(S_t)}\frac{A_{at}}{A_{mt}}\frac{K_{at}^{\alpha_1}}{K_{mt}^{\alpha_1}}\frac{E_{at}^{\alpha_3}}{E_{mt}^{\alpha_3}}\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-\alpha_2(\epsilon-1)}} \\ \frac{E_{at}}{E_{mt}} &= \left(\left(\frac{\tilde{w}_a}{\tilde{w}_m}\frac{\Omega_a(S_t)}{\Omega_m(S_t)}\frac{A_{at}}{A_{mt}}\frac{K_{at}^{\alpha_1}}{K_{mt}^{\alpha_1}}\frac{L_{at}^{\alpha_2}}{L_{mt}^{\alpha_2}}\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-\alpha_3(\epsilon-1)}}\end{aligned}$$

denote $\nu_1 \equiv \frac{\epsilon-1}{\epsilon-\alpha_1(\epsilon-1)}$, $\nu_2 \equiv \frac{\epsilon-1}{\epsilon-\alpha_2(\epsilon-1)}$ and $\nu_3 \equiv \frac{\epsilon-1}{\epsilon-\alpha_3(\epsilon-1)}$.

$$\frac{K_{at}}{K_{mt}} = \left(\frac{\tilde{w}_a}{\tilde{w}_m}\frac{\Omega_a(S_t)}{\Omega_m(S_t)}\frac{A_{at}}{A_{mt}}\right)^{\frac{\epsilon-1}{\epsilon-\alpha_1(\epsilon-1)}}\frac{L_{at}}{L_{mt}}^{\frac{\alpha_2(\epsilon-1)}{\epsilon-\alpha_1(\epsilon-1)}}\frac{E_{at}}{E_{mt}}^{\frac{\alpha_3(\epsilon-1)}{\epsilon-\alpha_1(\epsilon-1)}} \tag{55}$$

$$\frac{L_{at}}{L_{mt}} = \left(\frac{\tilde{w}_a}{\tilde{w}_m}\frac{\Omega_a(S_t)}{\Omega_m(S_t)}\frac{A_{at}}{A_{mt}}\right)^{\frac{\epsilon-1}{\epsilon-\alpha_2(\epsilon-1)}}\frac{K_{at}}{K_{mt}}^{\frac{\alpha_1(\epsilon-1)}{\epsilon-\alpha_2(\epsilon-1)}}\frac{E_{at}}{E_{mt}}^{\frac{\alpha_3(\epsilon-1)}{\epsilon-\alpha_2(\epsilon-1)}} \tag{56}$$

$$\frac{E_{at}}{E_{mt}} = \left(\frac{\tilde{w}_a}{\tilde{w}_m}\frac{\Omega_a(S_t)}{\Omega_m(S_t)}\frac{A_{at}}{A_{mt}}\right)^{\frac{\epsilon-1}{\epsilon-\alpha_3(\epsilon-1)}}\frac{K_{at}}{K_{mt}}^{\frac{\alpha_1(\epsilon-1)}{\epsilon-\alpha_3(\epsilon-1)}}\frac{L_{at}}{L_{mt}}^{\frac{\alpha_2(\epsilon-1)}{\epsilon-\alpha_3(\epsilon-1)}} \tag{57}$$

Substituting the labor share equation (56) into the capital share equation (55) and solving for the capital share gives us

$$\begin{aligned}\frac{K_{at}}{K_{mt}} &= \left(\frac{\tilde{w}_a}{\tilde{w}_m}\frac{\Omega_a(S_t)}{\Omega_m(S_t)}\frac{A_{at}}{A_{mt}}\right)^{\frac{\frac{\epsilon-1}{\epsilon-\alpha_1(\epsilon-1)}\left(1+\frac{\alpha_2(\epsilon-1)}{\epsilon-\alpha_2(\epsilon-1)}\right)}{1-\alpha_1\alpha_2\frac{\epsilon-1}{\epsilon-\alpha_1(\epsilon-1)}\frac{\epsilon-1}{\epsilon-\alpha_2(\epsilon-1)}}}\frac{E_{at}}{E_{mt}}^{\frac{\frac{\alpha_3(\epsilon-1)}{\epsilon-\alpha_1(\epsilon-1)}\left(1+\frac{\alpha_2(\epsilon-1)}{\epsilon-\alpha_2(\epsilon-1)}\right)}{1-\alpha_1\alpha_2\frac{\epsilon-1}{\epsilon-\alpha_1(\epsilon-1)}\frac{\epsilon-1}{\epsilon-\alpha_2(\epsilon-1)}}} \\ &= \left(\frac{\tilde{w}_a}{\tilde{w}_m}\frac{\Omega_a(S_t)}{\Omega_m(S_t)}\frac{A_{at}}{A_{mt}}\right)^{\frac{\epsilon-1}{\epsilon-\alpha_1(\epsilon-1)-\alpha_2(\epsilon-1)}}\frac{E_{at}}{E_{mt}}^{\frac{\alpha_3(\epsilon-1)}{\epsilon-\alpha_1(\epsilon-1)-\alpha_2(\epsilon-1)}}\end{aligned}$$

Now, substitute this into the labor share equation (56) to obtain

$$\begin{aligned}\frac{L_{at}}{L_{mt}} &= \left(\frac{\tilde{w}_a}{\tilde{w}_m}\frac{\Omega_a(S_t)}{\Omega_m(S_t)}\frac{A_{at}}{A_{mt}}\frac{E_{at}^{\alpha_3}}{E_{mt}^{\alpha_3}}\right)^{\frac{\epsilon-1}{\epsilon-\alpha_2(\epsilon-1)}+\frac{\alpha_1\frac{\epsilon-1}{\epsilon-\alpha_1(\epsilon-1)}\frac{\epsilon-1}{\epsilon-\alpha_2(\epsilon-1)}\left(1+\frac{\alpha_2(\epsilon-1)}{\epsilon-\alpha_2(\epsilon-1)}\right)}{1-\alpha_1\alpha_2\nu_1\nu_2}} \\ &= \left(\frac{\tilde{w}_a}{\tilde{w}_m}\frac{\Omega_a(S_t)}{\Omega_m(S_t)}\frac{A_{at}}{A_{mt}}\right)^{\frac{\epsilon-1}{\epsilon-\alpha_1(\epsilon-1)-\alpha_2(\epsilon-1)}}\frac{E_{at}}{E_{mt}}^{\frac{\alpha_3(\epsilon-1)}{\epsilon-\alpha_1(\epsilon-1)-\alpha_2(\epsilon-1)}}\end{aligned}$$

Substituting both these expressions into the emission share equation (57) we obtain after som algebra

$$\frac{E_{at}}{E_{mt}} = \left(\frac{\tilde{w}_a}{\tilde{w}_m} \frac{\Omega_a(S_t)}{\Omega_m(S_t)} \frac{A_{at}}{A_{mt}} \right)^{\frac{\epsilon-1}{\epsilon-(\epsilon-1)(\alpha_1+\alpha_2+\alpha_3)}}$$

Further, from (54) we have that $\frac{K_{at}}{K_{mt}} = \frac{L_{at}}{L_{mt}} = \frac{E_{at}}{E_{mt}}$ which implies that the solution will be equivalent for all factor inputs. Further it is clear that the above exponents simplifies to $\epsilon - 1$ if we assume constant returns to scale. ■

References

Acemoglu D. and Guerrieri V., 2008, Capital Deepening and Nonbalanced Economic Growth, *Journal of Political Economy*, vol. 116(3).

Archer D., 2005, Fate of fossil fuel CO_2 in geologic time, *Journal of Geophysical Research*, vol. 110(C9), pp. 1–6.

Arrow K.J., Chenery H.B., Minhas B.S. and Solow R.M., 1961, Capital-labor substitution and economic efficiency, *The Review of Economics and Statistics*, vol. 43(3), pp. 225–250.

Baumol W.J., 1967, Macroeconomics of Unbalanced Growth: The Anatomy of Urban Crisis, *American Economic Review*, vol. LVII, pp. 415–426.

Brock W.A. and Mirman L., 1972, Optimal Economic Growth and Uncertainty: The Discounted Case, *Journal of Economic Theory*, vol. 4(3), p. 479513.

Golosov M., Hassler J., Krusell P. and Aleh T., 2011, Optimal taxes on fossil fuel in general equilibrium, Working Paper 17348, National Bureau of Economic Research.

Hansen J.E., 2005, A slippery slope: How much global warming constitutes dangerous anthropogenic interference??, *Climatic Change*, vol. 68(3), pp. 269–279.

Hassler J., Krusell P. and Olovsson C., 2011, Energy Saving Technical Change, mimeo, Institute of International Economic Studies, Stockholm University.

Hoel M. and Sterner T., 2007, Discounting and relative prices, *Climatic Change*, vol. 84(3-4), pp. 265–280.

IPCC, 2001, *Climate Change 2001: The Third Assessment Report of the Intergovernmental Panel on Climate Change*, Cambridge University Press, Cambridge.

IPCC, 2007, *Climate Change 2007: The Fourth Assessment Report of the Intergovernmental Panel on Climate Change*, Cambridge University Press, Cambridge.

Irz X. and Roe T., 2005, Seeds of growth? Agricultural productivity and the transitional dynamics of the Ramsey model, *European Review of Agricultural Economics*, vol. 32(2), pp. 143–165.

Knorr W., 2009, Is the airborne fraction of anthropogenic CO_2 emissions increasing?, *Geophysical Research Letters*, vol. 36(L21710), pp. 1–5.

Lin J.Y. and Xu V.Z., 2011, The Hump-Shaped Structural Change, `http://space.uibe.edu.cn/u1/venite/my%20web/hump-shaped.pdf`.

Martin W. and Mitra D., 2001, Productivity Growth and Convergence in Agriculture versus Manufacturing, *Economic Development and Cultural Change*, vol. 49(2), pp. 403–422.

Ngai L.R. and Pissarides C.A., 2007, Structural Change in a Multisector Model of Growth, *American Economic Review*, vol. 7(1), pp. 429–43.

Nordhaus W. and Boyer J., 2000, *Warming the World: Economic Models of Global Warming*, MIT Press.

NORDHAUS W.D., 1994, *Managing the Global Commons: the economics of the greenhouse effect*, MIT Press, Cambridge, MA.

NORDHAUS W.D., 2007, A Question of Balance: Weighing the Options on Global Warming Policies, Tech. rep., Yale University.

ROCKSTRÖM J., STEFFEN W., NOONE K., PERSSON A., CHAPIN F.S., LAMBIN E.F., LENTON T., SCHEFFER M., FOLKE C., SCHELLNHUBER H.J., NYKVIST B., WIT C.A.D., HUGHES T., LEEUW S.V.D., RODHE H., SORLIN S., SNYDER P.K., COSTANZA R., SVEDIN U., FALKENMARK M., KARLBERG L., CORELL R.W., FABRY V.J., HANSEN J., WALKER B., LIVERMAN D., RICHARDSON K., CRUTZEN P. and FOLEY J.A., 2010, A safe operating space for humanity, *Nature*, vol. 461(7263), pp. 472–475.

ROE G.H. and BAKER M.B., 2007, Why is climate sensitivity so unpredictable?, *Science*, p. 318: 629632.

ROGNER H.H., 1997, An Assessment of World Hydrocarbon Resources, *Annual Review of Energy and the Environment*, vol. 22, pp. 217–262.

RUTTAN V.W., 2002, Productivity Growth in World Agriculture: Sources and Constraints, *Journal of Economic Perspectives*, vol. 16(4), pp. 161–184.

STANTON E.A., ACKERMAN F. and KARTHA S., 2009, Inside the integrated assessment models: Four issues in climate economics, *Climate and Development*, vol. 1, pp. 166–184.

STERNER T. and PERSSON M., 2008, An Even Sterner Review: Introducing Relative Prices into the Discounting Debate, *Review of Environmental Economics and Policy*, vol. 2(1), pp. 61–76.

TANS P., 2011, Trends in Carbon Dioxide, NOAA/ESRL, `http://www.esrl.noaa.gov/gmd/ccgg/trends/index.html`.

TIMMER C.P., 2009, A World without Agriculture? The Historical Paradox of Agricultural Development, AEI Development Policy Outlook series. 1, American Enterprise Institute for Public Policy Research.

TIMMER M.P. and DE VRIES G.J., 2009, Structural Change and Growth Accelerations in Asia and Latin America: A New Sectoral Data Set, *Cliometrica*, vol. 3(2), pp. 165–190.

TOL R.S.J., 2009, The Economic Effects of Climate Change, *Journal of Economic Perspectives*, vol. 23, pp. 29–51.

VALENTINYI A. and HERRENDORF B., 2008, Measuring factor income shares at the sectoral level, *Review of Economic Dynamics*, vol. 11(4), pp. 820 – 835.

WEITZMAN M.L., 2010, What Is the "Damages Function" for Global Warming and What Difference Might It Make?, *Climate Change Economics*, vol. 01(01).