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Climatic tipping points and optimal fossil fuel use*

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Abstract

The economics of climate change is characterized by many uncertainties, including climate dynamics, economic damages and potentially irreversible climate catastrophes. Using an optimal growth model of a fossil-fuel driven economy subject to a climate externality and potentially irreversible climatic events, this paper contributes to the understanding of how the risk of such events impacts on optimal fossil-fuel over time. Catastrophic events are modelled as irreversible abrupt changes in the underlying system dynamics. Our analytical results reveals the existence of three important effects concerning optimal fossil-fuel use; i) the existence of such events will increase the present value of marginal damages which works to postpone extraction ii) the probability of an event occurring sometime in the future also lowers the value of using fossil-fuels in the future which creates incentives to use more of the resource today iii) if the probability of a regime shift increases in fossil-fuel use this creates incentives to further postpone usage. Depending on the specification of the hazard rate process, which of the above effects dominates and the assumptions made regarding the abundance of fossil-fuel reserves, optimal extraction may become either increasingly precautionary or aggressive as

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a result of including potentially catastrophic events in the model.

Keywords: Catastrophic events, climate change, economic growth, hazard rate

1 Introduction

There is a growing concern among environmental and climate scientist's that the continuing increase in carbon dioxide emissions may trigger abrupt and possibly irreversible changes to the dynamics of the earth system (Alley et al., 2003; Lenton et al., 2008; Smith et al., 2009).[1] Despite this concern, most economist's addressing the climate-change dilemma using integrated assessment models (IAMs), typically abstract from such risks. The gravity of this neglect has recently been pointed out by Robert Pindyck who argues that IAMs are seriously flawed in a several distinct ways, one of them being the failure to properly account for potentially irreversible climate catastrophes (Pindyck, 2013a,b). This includes highly ambitious modeling attempts such as (Tol, 1999; Stern, 2006; Nordhaus, 2007) and more recently Golosov et al. (2014). In particular, since these models assume that the climate system evolves gradually they have little to say about how climate catastrophes impact on policy.

Recently, efforts have however been made to address this concern in IAMs. Examples include, Lemoine and Traeger (2013) and Cai et al. (2012) who analyze climate catastrophic events in an extended version of the well-known DICE model showing that the threat of a tipping point induces significant and immediate increases in the optimal carbon tax.[2] This is intuitively what one would expect since potential catastrophic events imply an increase in the expected marginal damages from climate change, but also since the continued injection

[1]Throughout this paper we will somewhat loosely refer to such events as regime shifts. Further, as in Lenton et al. (2008) we denote a "tipping point" as a critical threshold at which a tiny perturbation can qualitatively alter the state or development of a system and the term "tipping element" as a large-scale components of the Earth system that may pass a tipping point.

[2]See e.g. Nordhaus and Boyer (2000); Nordhaus (2007) for the "Dynamic Integrated Climate Economy" (DICE) model or the "Regional Integrated Climate Economy" (RICE) model.

of carbon dioxide into the atmosphere is likely to increase the probability of triggering such an event. Earlier work on the effects of introducing risks of regime shifts in models of climate change, pollution or renewable natural resources has however revealed that precautionary policies need not always result as a consequence of explicitly accounting for potential catastrophes in environmental pollution models (Cropper, 1976; Clarke and Reed, 1994; Tsur and Zemel, 1998; Polasky et al., 2011; Ren and Polasky, 2013; Quaas et al., 2013). In particular, whether the existence of catastrophic shifts implies a precautionary policy or not may depend on assumptions regarding risk-aversion, post-shift conditions and hazard rate functions. van der Ploeg and de Zeeuw (2013) have also shown, that there are competing effects involved when considering the risk of climate catastrophe events on optimal emission policy. In their model the existence of a potential tipping point also implies, a "be-prepared" effect which works via precautionary capital accumulation so as to better prepare when the disaster eventually hits. This effect induces an increase in fossil-fuel use and thus also results in a higher risk of catastrophe.

In this paper, we extend the above literature on tipping points in the climate economy context in several directions. First of all, as opposed to all previous modeling attempts we are aware of that accounts for potential catastrophic events in the climate-economy context, we will consider also the case when fossil fuels are finite in stock and essential in production.[3] This turns out to be a qualitatively important assumption. The reason for this is related to the mechanisms behind the "Green Paradox" (Sinn, 2008).[4] The possibility of a potential catastrophic and irreversible event appearing at some unknown date in the future, will in this case imply that the value of fossil-fuel use from that

[3]Whether the fossil-fuel stock should be considered a finite stock in this context is of course debatable, in particular given the huge new reserves of shale gas and other forms of unconventional gas and oil that are continuously being discovered. However, regardless of what opinion one may hold on this issue it is still becoming increasingly clear that fossil-fuel prices are rising and are expected to continue rising unless some new miracle substitute is by chance is discovered (Hassler et al., 2012; Golosov et al., 2014).

[4]The green paradox states that announcing that stringent emission policies will be introduced in the future, may induce oil producers to shift their production toward the present and thereby to exacerbate the problem of climate change.

point on will decrease, compared to a situation had the event not occurred. Hence, since the resource stock is finite, this implies that it may be optimal to use more of the stock prior to the event occurring since the value of preserving of fossil-fuel reserves for the future will then be lower. We will refer to this as the "Green Paradox effect". Secondly, we consider several potential kinds of tipping elements of the climate system by allowing all parameters of our climate module to shift. This also turns out to have qualitative impacts on policy. Here, we show that the type of tipping point considered can have, not only quantitative, but also qualitative impacts on the derived optimal policy and optimal carbon tax rate. The reason for this is that when there is a green paradox effect, the relative change of marginal damages caused by emissions at different points in time matter. Optimal fossil-fuel use at the outset may even increase if policy does not consider its impact on the probability of the event being triggered. Finally, in contrast to much of the above literature we derive several analytical proofs for many of the qualitative effects related to tipping points in relation to optimal policy, resorting to numerical measures only in few specific cases.

The model we develop is a discrete time standard macro economic growth model subject to a climate externality and a potential catastrophic and irreversible event. The structure of the model is closely related to the model derived by Golosov et al. (2014). As mentioned above, the model includes a potentially finite stock of fossil-fuel reserves. Our analysis considers four types of potential shocks to the climate system: a shift in the damages caused by a given stock of atmospheric carbon, a decrease in the immediate uptake of emissions by the biosphere, a sudden outburst of greenhouse gases and a decrease in the long-term uptake of carbon dioxide mainly by the oceans. Inline with earlier studies such as Cai et al. (2012); Lemoine and Traeger (2013) and van der Ploeg and de Zeeuw (2013) we model the risk of regime shifts using a hazard rate and distinguish between cases where the hazard rate is constant and cases where the hazard rate is endogenously determined as a function of previous emissions. We solve the model by working backwards, beginning with the post-tipping point solution. As mentioned above our main results are analytical and we need not resort to the use of advanced numerical techniques as found in e.g. Cai et al. (2012) and Lemoine and Traeger (2013). The study by van der Ploeg and de Zeeuw (2013) on the other hand, use a combination of analytic solutions in steady-state and numerical simulations for the paths when characterizing their results. The current paper thus contrasts to these studies since our main results are analytical and determined for the entire path. As previously mentioned our model build on the framework developed by Golosov et al. (2014). We thus make use of assumptions such as logarithmic utility, full capital depreciation and a Cobb-Douglas production function. These are strong assumptions which implies that our analysis does not let us consider how e.g. Epstein-Zin preferences (Cai et al., 2012) or precautionary capital accumulation (van der Ploeg and de Zeeuw, 2013) affect our results.

To sum up, this paper is about disentangling different effects involved when modeling tipping points in the climate system and how these effects come an go depending on how the tipping points are modeled. We identify three main interacting effects; i) the existence of a potential regime shift implies an increase in overall expected damages and hence creates incentives to postpone current fossil-fuel use to the future ii) the probability of a regime shift occurring sometime in the future however, also lowers the value of future fossil-fuel use and thus creates incentives to use more of the resource today (the Green Paradox effect) iii) with the probability of a regime shift increasing in fossil-fuel use this creates incentives to postpone fossil-fuel use (avert-risk effect). This implies that the impact of a potential tipping point on climate policy will depend on how tipping points and energy use is included in the model.

This paper is structured as follows. Section 2 describes the model details and how regime shifts or tipping events are modeled. Section 3 fully characterizes the solution to the post-regime shift problem. Section 4 characterizes the solution to the pre-regime shift point problem. Section 5 derives the analytical results w.r.t the qualitative behavior of optimal fossil-fuel management. Section 6 concludes.

2 Regime shifts in the climate-economy model

We begin by introducing the general features of the economy and how climate change impacts via both gradual changes and the risk of a regime shift in the climate system. The model we will develop is a standard neoclassical growth model in discrete time. Our focus will be on solving the planning problem involving optimal accumulation of capital and fossil fuel but reformulating the problem as a competitive equilibrium subject to fossil-fuel taxes would be straightforward.

2.1 General model features

We consider a representative consumer with a standard concave utility function $U(C)$ which is a function consumption C only. The consumer discounts utility between each period in time using a constant discount factor β.

The production process consists of a single aggregate commodity which is produced using capital K, L and energy E as input factors where capital and energy are endogenously determined variables while labor is in-elastically supplied. The production process is also subject to damages from climate change due to carbon dioxide accumulation in the atmosphere S which in turn is a direct result of emissions from fossil-fuel use in production. We write the aggregate production function as follows

$$Y = D(S)F(K, L, E), \tag{1}$$

where $D(S) \in [0, 1]$ denotes the damage function with $D'(S) < 0$ and $D''(S) > 0$. Given this production function capital dynamics are

$$K_{t+1} + C_t = Y_t + (1 - \delta_k)K_t,$$

where the left hand side denotes usage and the right hand side output plus undepreciated capital.

Energy use is modeled as directly proportional to fossil-fuel use which is extracted from a finite resource stock R_0. Given the vast amounts of fossil-fuel reserves and new resources such as tar sand etc. becoming available this is perhaps not a reasonable representation. We therefore consider also the case where resource stocks are infinite for sake of comparison. The general assumption will however be that resources are finite which implies that extraction over an infinite time horizon must obey

$$\sum_{t=0}^{\infty} E_t \leq R_0,$$

where R_0 denotes the initial stock of fossil fuels.

We have chosen not consider energy from carbon dioxide neutral sources or extraction costs in our model. This is fairly straightforward to include (for instance using the energy-sector model from Golosov et al. (2014)) but would come at a cost of increasing the complexity and readability of the analytical expressions we will derive below. In line with the purpose of the present paper, we have thus chosen to simplify for the reader and exclude this type of complexity, leaving it as an important area for future research.

Finally, concerning the atmospheric carbon dioxide stock we have chosen a fairly standard form for a stock pollutant which captures the stock as a function of past emissions and depriciation. Accumulation is modelled as

$$S_{t+1} = \sigma E_t + (1 - \delta) S_t. \tag{2}$$

This relationship is clearly an extreme simplification of the intricate process involving the carbon dioxide flow. This form has however been common in the climate economy literature since it apparently captures some fundamentals of the accumulation process involving immediate conversion of carbon dioxide from burning fossil fuel along with the long run decay. For our purposes here, this simple form will be sufficient since it contains sufficient complexity for us to make the point that it may matter a lot in terms of policy how different tipping points are considered within the model.

2.2 A few stylized assumptions

As in Golosov et al. (2014) the analytical tractability of our model relies upon some rather specific assumptions regarding some of the above functional forms and values for specific parameters. These are

$$U(C) = \ln(C) \tag{3a}$$

$$\delta_K = 1 \tag{3b}$$

$$F(K, L, E) = K^{\alpha} L^{1-\alpha-\nu} E^{\nu} \tag{3c}$$

$$D(S) = e^{-\gamma S}, \tag{3d}$$

where $\alpha > 0$ and $\nu > 0$ are constants such that $\alpha + \nu < 1$ and $\gamma \geq 0$ determines the severity of climate change induced productivity decreases. The validity of these assumptions may of course be questioned. First, a logarithmic utility may not be the preferred assumption regarding the elasticity of inter temporal substitution or risk aversion but it is commonly used and was e.g. standard in the early work on William Nordhaus's DICE and RICE models (Nordhaus, 1994; Nordhaus and Boyer, 2000). With a longer time period (we assume ten year periods) this curvature also becomes increasingly reasonable. Second, a completely depreciating capital stock in every time period (3b) is too high, even with a ten year time period. Golosov et al. (2014) have however, solved versions of their model while relaxing this assumption. This proved not to have any major effects. Third, concerning Cobb-Douglas production, Hassler et al. (2012) point out that, on shorter time horizons, this does not represent a good way of modeling energy demand since it does not capture the joint shorter- to medium-run movements of input prices and input shares. However, on the longer time scale we consider here it is more reasonable since input shares do not appear to trend over time. Finally, the exponential damage function differs from the standard quadratic damage functions used in e.g. the DICE and RICE models of Nordhaus. The calibration exercise in Golosov et al. (2014) has revealed that an exponential damage function of carbon dioxide can be calibrated to approximate a quadratic damage function of the global mean temperature fairly well. For further calibration and robustness checks for these assumptions see Barrage (2014).

2.3 Regime shifts

Potential regime shifts (or tipping points) are modeled as irreversible changes in the system dynamics which are expected to occur either with a constant probability (exogenous hazard rate) or endogenously determined probability (endogenous hazard rate) which is then a function of the carbon dioxide stock. There are several ways in which irreversible changes in the system dynamics may enter into the model. For example, van der Ploeg and de Zeeuw (2013) focus on tipping points as a potential productivity shock where a climate catastrophe is assumed to permanently reduce the productivity of the economy. Lemoine and Traeger (2013) model tipping points as permanent impacts on either the climate sensitivity parameter or the depreciation rate of atmospheric carbon dioxide. In this article our intentions are to take a broad approach to potential irreversible regime shifts, by considering changes in all model parameters that may exhibit tipping behavior in connection to the climate system of the model. As will be seen, the qualitative impacts on the optimal fossil-fuel use policy will differ depending on which parameters are assumed to change. From the model specification above, the parameters that can exhibit potential tipping behavior include $\{\gamma, \sigma, \delta\}$ which we will refer to as the tipping elements of the model (see Lenton et al. (2008)). Apart from these parameters we will also consider the possibility for a sudden release or a pulse of methane being emitted into the atmosphere, perhaps as a result of melting ice or permafrost (for permafrost related tipping points see e.g. Schaefer et al. (2011)). We will denote such a pulse of carbon dioxide equivalents by $\hat{P}$. The assumption is that such a pulse adds to the climate state S and then decays along with carbon dioxide in the general form of the carbon cycle. In reality, the dynamics of methane would be different from that of carbon dioxide, e.g. causing more damages in the short run but decaying faster. However, as long as it is assumed to follow a qualitatively similar decay process, expressing the methane pulse as an equivalent pulse of carbon dioxide will not affect our results and we make this convenient assumption to save on notation.

Next, since these parameters shift discontinuously, in the post-regime shift era, we will denote parameters using a ˆ notation so that $\{\hat{\gamma}, \hat{\sigma}, \hat{\delta}, \hat{\hat{P}}\}$ denote values after the regime shift has occurred.[5] We also assume that the shift implies the situation for society is worse than prior to the shift which implies that these new set of parameters are all larger than their preshift counterparts with the exception of $\hat{\delta}$ which is smaller than δ implying that the long term uptake of carbon dioxide decreases after the shift.

We thus have four different types of regime shifts that could happen representing different types of changes in the climate system and the climate-economy interaction. An increase in γ represents an increase in damages from any given amount of climate change (as measured by S). This means that the consequences of any caused amount of change will be more severe. The damages do not change until the shift happens but since current emissions affect the climate also in the future, the total marginal damages increase for emissions made before the shift as well. An increase in σ represents a decrease in the the share of emissions that are taken up immediately. This does not affect the damages caused by emissions made prior to the shift (except through potentially increasing the risk of a shift). A decrease in δ represents a decrease in the decay rate of atmospheric carbon. Since this affects also the decay rate of carbon that was emitted prior to the shift, this affects marginal damages caused by emissions made prior to the shift. A methane pulse $\hat{P}$ does not, under the assumptions made, affect the marginal damages of further emissions and does thus only affect optimal amount of fossil fuel-use through its effect on the probability of the shift occurring. Despite them representing very different types of shifts, it will

[5]Note that the greenhouse-gas pulse $\hat{P}$ is assumed to be zero prior to the shift.

turn out below that the effects of potential shifts in γ and δ will be qualitatively similar. This can be understood since they both affect damages of emissions made prior to the shift through the persistence of climate change (but marginal damages increase more for post-shift emissions compared to pre-shift emissions).

Finally, before proceeding to solve the model we also need to characterize the uncertainty. Let T denote a discrete random variable indicating the time of a regime shift occurring. We define the probability of a regime shift occurring, the hazard rate, as

$$\pi_t = \Pr(T = t | T \geq t). \tag{4}$$

The hazard rate is thus the conditional probability of a regime shift occurring in time period t given it has not occurred in previous time periods. The unconditional probability of an event occurring at time t is thus given by the following probability mass function $\Pr(T = t) = \pi_t \prod_{s=0}^{t-1}(1 - \pi_s)$ and the probability of an event not occuring before time t (the survival function) is $\Pr(T > t) = \prod_{s=0}^{t}(1 - \pi_s)$. More generally, we define the survival probability

$$\Omega_{n_1}^{n_2} = \begin{cases} \prod_{m=n_1}^{n_2-1}(1 - \pi_m) & \text{if } n_1 < n_2 \\ 1 & \text{otherwise} \end{cases}. \tag{5}$$

This is the probability that there will not be any regime shift before period n_2 conditional on that it has not occurred before n_1. We will consider both the case where the hazard rate is constant over time but also the case where the hazard rate is endogenously determined by the carbon dioxide stock in the atmosphere, i.e., we have $\pi_t = \pi(S_t)$. In this case we will assume that $\partial\pi(S)/\partial S > 0$. The probability density function is then

$$P(T = t) = \pi(S_T) \prod_{t=0}^{t-1}(1 - \pi(S_t)). \tag{6}$$

We will now formulate the planner solution of this model. The solution to this problem corresponds to the optimal allocation and the optimal tax in a decentralized equilibrium with taxation of fossil-fuel use can easily be derived from this solution.

2.4 Planning problem

The planning problem can thus be written as

$$
\begin{aligned}
V(K_0, R_0, S_0) = & \max_{\substack{\{C_t, E_t, K_{t+1}, S_{t+1}, R_{t+1} \\ K_T, S_T, R_T\}_{t=0}^{T-1}}} \mathbb{E}_0 \sum_{t=0}^{T-1} \beta^t U(C_t) + \beta^T \hat{V}(K_T, R_T, S_T + \hat{P}) \\
\text{s.t. } & K_{t+1} = D(S_t) F(K_t, E_t) - C_t + (1 - \delta_k) K_t \\
& S_{t+1} = \sigma E_t + (1 - \delta) S_t \\
& R_{t+1} = R_t - E_t \\
& \sum_{t=0}^{\infty} E_t \leq R_0,
\end{aligned}
$$

where T is a random variable with distribution given by (6). Here $\hat{V}(K_T, R_T, S_T + \hat{P})$ denotes the maximized value function of the post-shift problem. In order to solve this we use dynamic programming and start with the solution to the post-shift problem and thereafter proceed with the pre-shift solution.

3 Post-shift solution

Given the assumption of an irreversible tipping point, once the tipping has occurred there is no going back and the problem from that point and onwards is deterministic and all parameters will be constants. We denote the post regime shift value function by $\hat{V}$.

We will solve the optimization problem using dynamic programming with state variables capital, K, remaining fossil-fuel resources, R, and the amount of carbon in the atmosphere S. We follow standard dynamic programming notation so that for any variable X, X' refers to the next periods value for that variable.[6]

As control variables we choose next periods capital stock K' and the current period fossil-fuel use E. Based on the model components described in section 2.1, we have the following relationships[7]

$$\begin{aligned} C &= \hat{D}(S)F(K,E) - K' \\ R' &= R - E \\ S' &= \hat{\sigma}E + (1-\hat{\delta})S \end{aligned}$$

and the Bellman equation of the planner problem is

$$\hat{V}(K,R,S) = \max_{K',E} \left[\begin{array}{c} U\left(\hat{D}(S)F(K,E) - K'\right) \\ +\beta\hat{V}\left(K', R-E, \hat{\sigma}E + (1-\hat{\delta})S\right) \end{array} \right]. \tag{7}$$

Solving this problem (see appendix A for details) we obtain the optimal decision rules that are given by the constant savings rate

$$C = (1-\alpha\beta)Y \text{ and } K' = \alpha\beta Y. \tag{8}$$

and fossil-fuel use

$$E = \frac{\nu}{(1-\alpha\beta)\left(\hat{V}_R(K,R,S) + \beta\hat{\sigma}\hat{\Gamma}\right)}. \tag{9}$$

where $\hat{V}_R(K,R,S)$ is the shadow value of the fossil-fuel resources and $\hat{\Gamma}$ is the constant shadow value of the stock of atmospheric CO_2

$$\hat{\Gamma} = \frac{\hat{\gamma}}{1-\alpha\beta}\frac{1}{1-\beta(1-\hat{\delta})}$$

[6]This applies only to variables. In the case of general functional forms of one variable, the prime represents the first order derivative (e.g. $U'(C) = dU/dC$). This is also standard notation and should hopefully not cause confusion. Partial derivatives will be denoted by subscripts w.r.t. to the differentiation variable e.g. $F_K(K,L) = \partial F(K,L)/\partial K$.

[7]The damage function $\hat{D}$ is parametrized by $\hat{\gamma}$. If the tipping point involves a greenhouse-gas pulse $\hat{P}$, this will already be part of the climate state S and will not explicitly show up in the post regime shift solution.

capturing the discounted future damages caused by one unit of atmospheric CO_2.

From (3) we can immediately see that marginal damages increases in $\hat{\gamma}$ (a given amount of atmospheric CO_2 causes more damages) and decreases in $\hat{\delta}$ (decreased depreciation rate means that CO_2 stays for longer time in the atmosphere)

$$\frac{\partial \hat{\Gamma}}{\partial \hat{\gamma}} > 0, \ \frac{\partial \hat{\Gamma}}{\partial \hat{\delta}} < 0. \tag{10}$$

The shadow value of fossil fuel resources must, along the optimal path, increase at the rate $\frac{1}{\beta}$

$$\hat{V}_R(K, R, S) = \beta \hat{V}_R(K', R', S').$$

Combining this with the condition for fossil-fuel use (9) and the (binding) resource constraint on total available amounts of fossil fuel, we get that

$$R = \frac{\nu}{1 - \alpha\beta} \sum_{n=0}^{\infty} \frac{1}{\beta^{-n} \hat{V}_R(K, R, S) + \beta \hat{\sigma} \hat{\Gamma}}.$$

From this we can see that $\hat{V}_R$ depends only on R (and not not K or S). The post-shift value function is additively separable and is given by

$$\hat{V}(K, R, S) = \frac{\alpha}{1 - \alpha\beta} \ln K + \hat{W}(R) - \hat{\Gamma} S. \tag{11}$$

where $\hat{W}(R)$ is a strictly increasing and strictly concave function that fulfills

$$R = \frac{\nu}{1 - \alpha\beta} \sum_{n=0}^{\infty} \frac{1}{\beta^{-n} \hat{W}'(R) + \beta \hat{\sigma} \hat{\Gamma}}. \tag{12}$$

From (12) we get the following proposition:

Proposition 1. The shadow value of the fossil-fuel resource after the regime shift has happened depends only on parameters and R. The effects of changes in parameters and R on the shadow value are given by the inequalities

$$\frac{\partial \hat{V}_R}{\partial R} < 0, \; \frac{\partial \hat{V}_R}{\partial \hat{\sigma}} < 0, \; \frac{\partial \hat{V}_R}{\partial \hat{\delta}} > 0 \text{ and } \frac{\partial \hat{V}_R}{\partial \hat{\gamma}} < 0.$$

Proof. Follows from (12) and (10) with $\hat{W}'(R) = \hat{V}_R$. □

Summing up, starting from some state variables K, R, and S after the regime shift has occurred, the entire future path is determined by the constant consumption/savings rule (8) and the fossil-fuel use condition (9) where $\hat{V}_R(K, R, S) = \hat{W}'(R)$ fulfills (12). The value function is given by (11). We now move backwards in time to the situation before the regime shift has occured.

4 Pre-shift solution

We denote the value function before the regime shift by $V(K, R, S)$. The probability of a regime shift between the current and the next period is $\pi(S')$, that is, it depends on next period climate state S'. To simplify the notation we set

$$V_X\left(K^{(n)}, R^{(n)}, S^{(n)}\right) \equiv V_X^{(n)} \text{ and } \hat{V}_X\left(K^{(n)}, R^{(n)}, S^{(n)} + \hat{P}\right) \equiv \hat{V}_X^{(n)}$$

where (n) denotes the variabels' values n periods into the future conditional on that the regime shift has not happened yet for V and that it happens n periods into the future for $\hat{V}$. Using this notation, the Bellman equation can be written as

$$V(K, R, S) = \max_{K', E}\left(U(C) + \beta\left[\left(1 - \pi(S')\right)V' + \pi(S')\hat{V}'\right]\right) \tag{13}$$

with

$$\begin{aligned} C &= D(S)F(K, E) - K' \\ R' &= R - E \\ S' &= \sigma E + (1 - \delta)S. \end{aligned}$$

We go through the computations of the pre-shift problem in detail in appendix B.

The first-order condition with respect to K' gives us

$$U'(C) = \beta\left[\left(1-\pi(S')\right)V_K' + \pi(S')\hat{V}_K'\right]. \tag{14}$$

That is, the value of current consumption is equal to the expected value of next periods marginal value of holding capital. Using the envelope condition to get expressions for the shadow value of next period capital and applying our specific functional forms, we can derive the same savings rate as in the post-shift problem. The consumption/investment rule is thus given by (8) in the pre-shift problem as well.

Furthermore, capital will enter as a separable term in the value function which can be written as

$$V(K,S,R) = \frac{\alpha}{1-\alpha\beta}\ln K + W(R,S) \tag{15}$$

Turning instead to the first-order condition with respect to E we have

$$\begin{aligned} U'(C)D(S)F_E(K,E) \quad &= \quad \beta\left[(1-\pi(S'))V_R' + \pi(S')\hat{V}_R'\right] \\ &\quad -\beta\sigma\left[(1-\pi(S'))V_S' + \pi(S')\hat{V}_S'\right] \\ &\quad \beta\sigma\pi'(S')\left(V' - \hat{V}'\right). \end{aligned} \tag{16}$$

The left-hand side is the marginal value of using fossil fuel in production while the right-hand side consists of three different aspects representing costs associated with fossil-fuel use that the benefits should be traded off against. The first aspect is next period's expected shadow value of the resource stock. The second aspect is the expected next-period shadow-value of the climate state. Here, a regime shift increases the negative consequences of emissions which reduces the value of fossil-fuel use today. The third aspect is the effect of emissions on the probability of regime shift. Since, fossil-fuel use increases the likelihood of a shift this reduces our incentives to use fossil fuels today. This is the avert-risk affect.

By substituting forwards in the first-order condition we can derive the rule for optimal fossil-fuel use

$$\frac{\nu}{1-\alpha\beta}\frac{1}{E} = V_R + \Theta + \beta\sigma\tilde{\Gamma}, \tag{17}$$

where

$$\Theta = \beta\sigma \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \pi' \left(S^{(n+1)}\right) \left(V^{(n+1)} - \hat{V}^{(n+1)}\right) \Omega_1^{n+1} \tag{18}$$

and

$$\begin{aligned} \tilde{\Gamma}\left(\{S^{(l)}\}\right) &= \frac{\gamma}{1-\alpha\beta} \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \left(1 - \pi\left(S^{(n+1)}\right)\right) \Omega_1^{n+1} \\ &\quad + \hat{\Gamma} \sum_{n=0}^{\infty} \pi\left(S^{(n+1)}\right) \beta^n (1-\delta)^n \Omega_1^{n+1}. \end{aligned} \tag{19}$$

The factor Ω_1^{n+1} is the survival probability defined in (5). In the first-order condition (17), V_R gives the shadow value of the resource, Θ captures the effect of fossil-fuel use on the probability of the regime shift and $\tilde{\Gamma}$ captures the effect on expected marginal damage caused by fossil-fuel use.

We can thus separate out the three different aspects determining the amount of optimal fossil-fuel use in a given period. These can be considered separately. We start by considering the effect on Γ, that is the expected marginal damages excluding effects on the probability of a shift occurring. We can define

$$\Gamma = \frac{\gamma}{1-\alpha\beta}\frac{1}{\beta(1-\delta)}$$

which is equual to $\tilde{\Gamma}$ when $\pi = 0$ and gives the marginal damages if there is no risk of a shift. Comparing Γ, $\tilde{\Gamma}$ and $\hat{\Gamma}$, we have the following proposition

Proposition 2. If $\hat{\gamma} = \gamma$ and $\hat{\delta} = \delta$, then

$$\Gamma = \hat{\Gamma} = \tilde{\Gamma}\left(\{S^{(l)}\}\right)$$

for any sequence $\{S^{(l)}\}$. If instead $\hat{\gamma} \geq \gamma$ and $\hat{\delta} \leq \delta$ with at least one strict inequality, then

$$\Gamma < \hat{\Gamma} \text{ and } \tilde{\Gamma}\left(\{S^{(l)}\}\right) \in \left(\Gamma, \hat{\Gamma}\right)$$

for any sequence $\{S^{(l)}\}$ such that $\Omega_1^n \in (0,1)$ for some n.

Proof. See appendix C. □

Introducing the risk of a shift that affects either γ or δ thus increases the marginal damages. The expected pre-shift marginal damages lie between the deterministic marginal damages in the case without a risk of shift and post-shift in the case with a risk of a shift.

The part of the marginal effects of emissions that capture the increased risk of a shift depends on the difference between the pre-shift and post-shift value function. Regarding these, we have the following proposition:

Proposition 3. Assuming that $\hat{\sigma} \geq \sigma$, $\hat{\gamma} \geq \gamma$, $\hat{\delta} \leq \delta$ and $\hat{P} \geq 0$, then for any set of state variables (K, R, S) we have that

$$V(K, R, S) \geq \hat{V}(K, R, S + \hat{P})$$

and the inequality is strict if at least one of the parameter inequalities are strict.

Proof. See appendix D. □

The sign of Θ follows immediately as stated in the following proposition

Proposition 4. Assume that $\hat{\sigma} \geq \sigma$, $\hat{\gamma} \geq \gamma$, $\hat{\delta} \leq \delta$ and $\hat{\hat{P}} \geq 0$, with at least one strict inequality. Assuming, furthermore, that there is a strictly positive probability of regime shift in some future period and that $\pi'(S) > 0$ in at least one such period, then

$$\Theta > 0.$$

If, instead $\pi'(S) = 0$ for all S, then $\Theta = 0$.

Proof. The inequality for Θ follows since each term in the right-hand side of (18) is (weakly) positive and under the given assumptions at least one of them is strictly positive. □

The part of the damages that depend on the risk of triggering a regime shift is thus weakly positive and strictly positive whenever emissions affect the probability of a shift that has negative welfare effects.

The third term in (17), V_R, is the shadow value of the fossil-fuel resources. A higher shadow value translates into less fossil-fuel use. As shown below, the introduction of a potential regime shift decreases the shadow value.

In general all three terms in (17) will depend on the entire future path and we will now analyze the net effects of changes in all of these variables on optimal fossil-fuel use.

5 Effects of regime shift on fossil-fuel use

From the previous section we know that introducing the potential for a regime shift into the model will have three different qualitative effects on optimal fossil fuel use. The marginal damages caused by emissions will increase (weakly). Secondly, emissions (weakly) increase the risk of the regime shift happening. Both of these changes will tend to decrease fossil-fuel use. These changes in the attractiveness of using fossil fuel use will also change the shadow value of the fossil-fuel resources. This will cause an effect going in the opposite direction and the net effect will depend on various factors as described below. We start by analysing the case where the constraint on the total availability of fossil fuel is not binding. In that case we have that the shadow value, V_R, is equal to zero and we do then not have the third effect. We will then analyze the case where the constraint is binding.

5.1 Without resource scarcity

We will here consider the case where the constraint on the total amount of fossil fuel is not binding. Without a binding constraint on the total amount of fossil-fuel resources, $V_R = \hat{V}_R = 0$ in all time periods.[8]

Consider first how the post-regime shift environment is affected by the different tipping elements compared to if they had not existed. By equation (29) we can see that any tipping point such that $\hat{\sigma} > \sigma$, $\hat{\gamma} > \gamma$ or $\hat{\delta} < \delta$ will imply a decrease in the post-regime shift fossil-fuel use while the threat of a greenhouse gas pulse $\hat{P} > 0$ will not have any effect on fossil-fuel use even though the post-regime shift welfare level will be affected.

Next, consider the pre-regime shift environment. Here, optimal fossil-fuel use is determined by equation (17). The effect of introducing a regime shift on fossil-fuel use before the shift has occurred can be seen almost directly from propositions 2 and 3. Consider first the case when the probability of a regime shift is exogenous, i.e. $\pi'(S) = 0$ for all S. Then proposition 3 implies that $\Theta = 0$ and that the effect of a potential regime shift on fossil-fuel use before the event has occurred will only be determined by the effect it has on $\tilde{\Gamma}$. We can thus state the following proposition

Proposition 5. With no resource scarcity and an exogenous probability of a shift we have that $V_R = \Theta = 0$. A regime shift that affects $\hat{\sigma}$ or $\hat{P}$ will affect neither $\tilde{\Gamma}$ nor the pre-shift fossil-fuel use while an increase in $\hat{\gamma}$ or a decrease in $\hat{\delta}$ will increase $\tilde{\Gamma}$ and decreases the pre-shift fossil-fuel use.

Proof. Follows from proposition 2 and equation (17). □

The consequences of proposition 5 are depicted graphically in figure 1. Here the solid lines denote optimal fossil-fuel use without the threat of a regime shift while the dashed lines depict potential optimal fossil fuel paths with a potential regime shift present. As can be seen and is stated in the proposition only regime

[8]We could do this more formally by using the model for coal extraction in Golosov et al. (2014), but just setting the shadow value of the resources to zero is sufficient for present purposes.

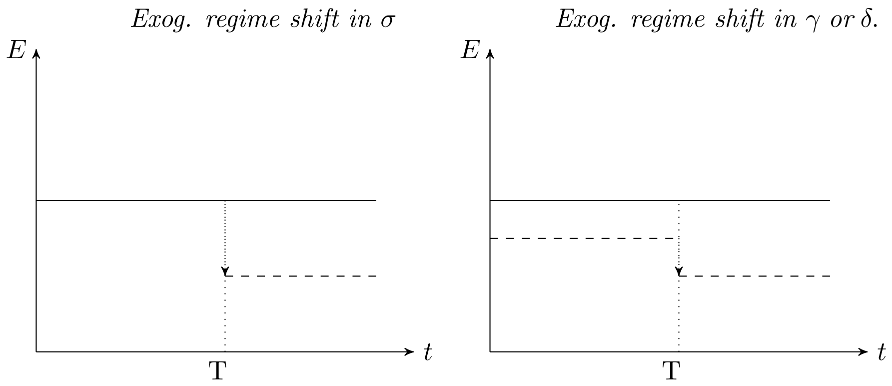

Figure 1: Examples of effects on optimal fossil-fuel use with infinite resources and an exogenous probability of a regime shift. The left hand graph shows effects of regime shifts acting on σ such that $\sigma < \hat{\sigma}$ while the right hand graph shows the impact of either γ or δ such that $\gamma < \hat{\gamma}$ or $\delta > \hat{\delta}$. The solid line in both graphs shows optimal fossil-fuel use without a regime shift while the dashed lines shows how these paths are altered when a regime shift (T) takes place.

shifts such that $\gamma < \hat{\gamma}$ or $\delta > \hat{\delta}$ imply precautionary behavior while $\sigma < \hat{\sigma}$ only reduces post regime shift use. This can be seen by comparing the left-hand graph to that on the right hand. Here, T denotes an arbitrary point in time when the regime shift occurs. As can be seen from the left-hand graph (illustrating a shift in $\sigma < \hat{\sigma}$) it is only after this event has occurred that the optimal fossil-fuel use is reduced. Prior to the event optimal fossil-fuel use is the same as it is when there is no threat (solid line). A shift causing a greenhouse-gas pulse, $\hat{P} > 0$, on the other hand has no effect at all on optimal policy and fossil-fuel use is thus the same as it is without a regime shift present. The right-hand graph (illustrating a shift in either $\gamma < \hat{\gamma}$ or $\delta > \hat{\delta}$) features a two-step decrease in fossil-fuel use. The first concerns the pre-shift fossil-fuel use while the second (post T) involves the post-regime shift further reduction in fossil-fuel use.

Turning now to the case when the probability of tipping is endogenous. By proposition 3 this implies that any kind of regime shift will make Θ positive and as shown in the proposition (6) below this implies a decrease the pre-shift fossil-fuel use in all cases.

Proposition 6. With no resource scarcity but with an endogenous probability of a shift, a potential regime shift of any kind will decrease fossil-fuel use before regime shift.

Proof. Proposition 5 implies that $\tilde{\Gamma}$ increases weakly and since the endogenity of the regime shift implies $\Theta > 0$, (17) gives us that pre-shift fossil-fuel use will decrease strictly. □

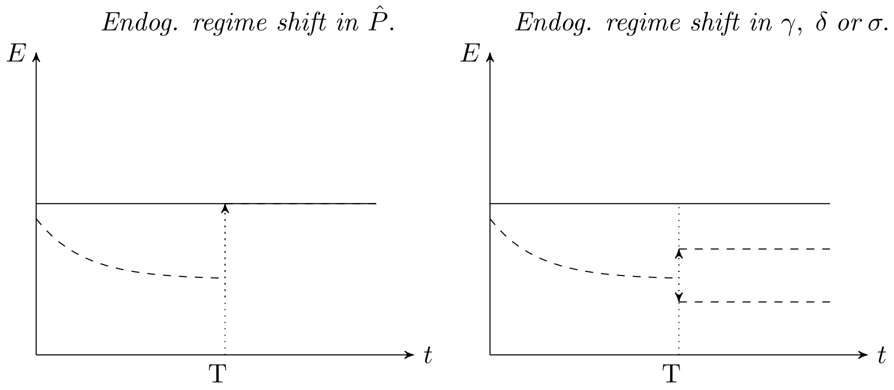

Figure 2: Examples of effects on optimal fossil-fuel use with infinite resources and an endogenous probability of a regime shift. The left-hand graph shows potential effects of regime shifts acting on σ or $\hat{P}$ such that $\sigma < \hat{\sigma}$ or $\hat{P} > 0$ while the right-hand graph shows the impact of either γ or δ such that $\gamma < \hat{\gamma}$ or $\delta > \hat{\delta}$. The solid line in both graphs shows optimal fossil-fuel use without a regime shift while the dashed lines shows how these paths are altered when a regime shift (T) takes place.

The main effects of the regime-shift can be seen in figure 2. Here, we see that prior to the regime shift fossil-fuel use is lower compared to the case without a regime shift ($t < T$). The path will change non-linearly due to the effect of changes in probability of a regime shift Θ. In general one cannot say anything regarding the slope of the path since it will depend on the shape of the hazard rate function and it may be positive or negative and change non-linearly. Eventually, this path will however approach its steady state given that no regime shift has occurred before then. Furthermore, it is also ambiguous what will happen when a shift finally occurs. It may be the case that the pre-shift fossil-fuel use is actually below the post-shift optimal fossil-fuel use for a given R due to a strong avert-risk effect. If a shift occurs at this point optimal fossil fuel use would thus shift up in the post-regime shift solution compared to the pre-shift solution. Also the opposite may be true. In table 4 in appendix we display parameter values that exhibit captures this behavior.

The main results of this section are outlined in table 1. Here, the first two columns denote the effect on optimal management prior to the regime shift for exogenous (π) and endogenous $\pi(S)$ probabilities while the second two columns denote the difference in post-shift fossil-fuel use compared to pre-shift fossil-fuel use. It is important to note that the first two columns portray the effects under the potential threat of a regime shift while the second columns denote the effect in relation to the pre-regime path. The first two columns thus involve decision making under uncertainty while the second columns involve decision making once the uncertainty has been resolved. In the post-shift columns the words "Increase", "Decrease", "No effect" and "Ambiguous" refer to the impact on optimal fossil-fuel use once the shift has occurred. That is when a regime-shift occurs the jump in the policy variable E can be increasing or decreasing but also unaffected. Further, when we write "Ambiguous" this implies that the outcome depends on the chosen parameter estimates or functional forms. Meanwhile, in the pre-shift environment the policy outcome of a potential shift in a specific parameter implies that policy may become precautionary or aggresive in comparison to optimal policy if no such risk of a shift was present. However, in this case we may also have that optimal policy is unaffected (as can be seen from the table this is the case for e.g. $\hat{P}$ when the hazard rate is exogenous)). As can be seen from the table the general pre-regime shift result is precautionary although there exist exceptions as described above.

We have thus cleared up most of the effects of a potential regime shift on optimal fossil-fuel when resource stocks are infinite and can now turn to the case when resource stocks are finite.[9]

[9]Note, that if we study the combined effect of a shift in e.g. both γ and $\hat{P}$ then $\hat{P}$ will act to increase the impact of the change in γ on initial fossil-fuel use when the probability of a regime shift is endogenous.

	Pre-regime shift		Post-regime shift	
	π	$\pi(S)$	π	$\pi(S)$
σ	No effect	Precautionary	Decrease	Ambiguous
$\hat{P}$	No effect	Precautionary	No effect	Increase
γ	Precautionary	Precautionary	Decrease	Ambiguous
δ	Precautionary	Precautionary	Decrease	Ambiguous

Table 1: Without resource scarcity: Effects of a potential regime shift threat on optimal fossil-fuel use for the four potential tipping elements $\{\sigma, \hat{P}, \gamma, \delta\}$.

5.2 With resource scarcity

When the resource is scarce, we must also consider the effect of introducing a potential regime shift on the scarcity value of fossil-fuel resources, V_R in (17). From propositions 2 and 3 we know that without resource scarcity a potential regime shift weakly decreases fossil-fuel use in all periods with strict decreases in some periods. Since this means that fossil-fuel use becomes less valuable, it seems reasonable that the introduction of a potential regime shift decreases the shadow value of the scarce resource. In the proposition below we show that this is the case implying that we will always have a Green Paradox type effect counteracting the tendency for a decrease of fossil-fuel use which comes from increasing marginal damages.

Proposition 7. Assume that $\hat{\delta} \leq \delta$, $\hat{\gamma} \geq \gamma$, $\hat{\sigma} \geq \sigma$ and $\hat{P} \geq 0$. Furthermore, assume that either at least one of the first three inequalities is strict or, if all the first three inequalities are equalities, the fourth inequality is strict and the regime shift is endogenous with $\pi'(S) > 0$ for all S. Then, for given state variables (K, R, S) and assuming that the regime shift has not happened, V_R will be smaller when there is a potential regime shift compared to a situation where there can not be any regime shift.

Proof. See appendix F. □

What this proposition says is that introducing the possibility of a regime shift tends to decrease fossil-fuel use but that this tendency is counteracted by a Green Paradox effect pulling in the opposite direction. The net effect in a particular period will thus depend on which effect is the strongest. The Green paradox effect compensates for the other two effects (both implying weakly lower fossil-fuel use in all periods) so that introducing the potential regime shift into the model does not affect the accumulated fossil-fuel use. The net effect in a given time period will thus depend on whether the strength of the other two effects is stronger or weaker in the time period under consideration compared to other time periods. In general determining this requires analyzing the entire path.

One case where we can unambiguously say what happens in the first period is when the tipping point only changes the value of σ to $\hat{\sigma} > \sigma$ and where the probability of tipping is exogenous.

Proposition 8. If resources are scarce and there is a constant probability of a regime shift working only through σ so that $\hat{\sigma} > \sigma$, then the Green Paradox effect will dominate leading to an increase in initial fossil-fuel extraction.

Proof. We begin by noting that since $\Theta = 0$ and $\tilde{\Gamma} = \Gamma$, by equation (17), pre-shift extraction can be written as.

$$E = \frac{\nu}{1 - \alpha\beta} \frac{1}{V_R + \beta\sigma\Gamma}$$

By proposition (7) we further know that V_R must initially be smaller due to the regime shift. Hence since there is no impact on the expected present value of marginal damages initial fossil-fuel use will thus be larger due to the regime shift. □

By proposition (8) we thus see that if the expected present value of marginal damages in a given period is unaffected by the potential regime shift (while it has a striclty positive effect on the damages cause by emissions in some other period) this implies that the Green Paradox effect will dominate leading to an increase in initial fossil-fuel use compared to the no-regime shift case. In all other cases we will have a combination of effects and the net result will in general, be ambiguous. An exception is for the case of a methane release $\hat{P}$. Here we have that when the probability of a regime shift is constant, there will be no effect on optimal policy. To see this note that neither the expected present value of marginal damages $\tilde{\Gamma}$ nor the marginal value of the resource stock V_R in the pre- and post- regime shift environment is affected by a sudden burst of $\hat{P}$.

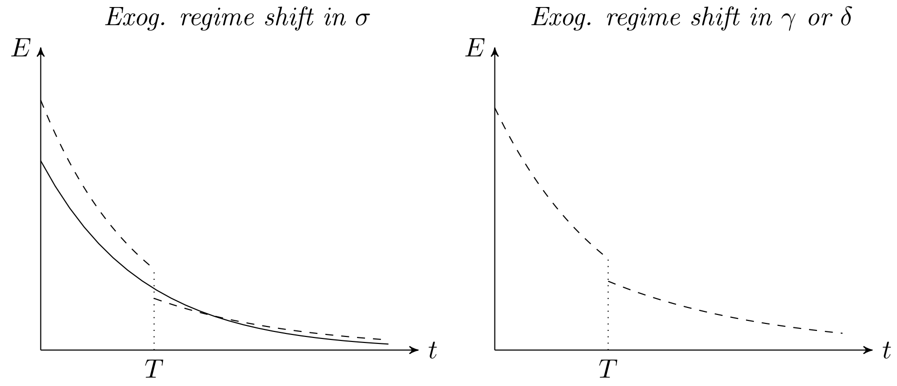

Figure 3: Examples of qualitative effects on optimal fossil-fuel use with finite resources and an exogenous probability of a regime shift. As above the dashed lines denote optimal paths with a potential regime shift and the solid line denotes the no-regime shift case for comparison.

The implications of proposition (8) can also be seen in figure 3. In the left-hand graph we see the dominating Green Paradox effect that results from the regime shift occurring in the tipping element σ which by proposition (8) thus implies a aggressive policy towards fossil-fuel use. The right hand graph portrays the effects of the tipping elements $\{\gamma, \delta\}$. Here, we have not depicted the no-regime shift fossil-fuel path. This is due to the fact that we cannot determine under which circumstances one the Green Paradox effect may dominate the marginal damage effect. This will instead depend on the choices of model parameter. We can however say something with respect to how a post-regime shift management policy compares to a pre-regime shift policy for a given stock of R and S i.e. what determines the shift at T in figure 3.

Proposition 9. Given that a regime shift occurs at some period $T > 0$ in one of the tipping elements $\{\sigma, \gamma, \delta\}$ (see figure 3 as an example), fossil-fuel use will be lower for a given R under the post-regime shift management policy compared to optimal policy had the regime shift not yet occurred.

Proof. To see this we first note that by proposition 2 we have that either $\tilde{\Gamma}(S) < \hat{\Gamma}$ for shifts occurring in $\{\gamma, \delta\}$ or that $\sigma < \hat{\sigma}$. In all cases we have that $\sigma\tilde{\Gamma}(S) < \hat{\sigma}\hat{\Gamma}$.

Assume now that $V_R' + \beta\sigma\tilde{\Gamma} > \hat{V}_R' + \beta\hat{\sigma}\hat{\Gamma}$. Then given proposition 12 we also have that $V_R^{(n)} + \beta\sigma\tilde{\Gamma} > \hat{V}_R^{(n)} + \beta\hat{\sigma}\hat{\Gamma}$ for all succeeding periods. This lead to a contradiction since in this case the sum of all future fossil-fuel use would be larger in the post-regime shift case which can not be true since total fossil-fuel use must be the same in both cases. □

	Pre-regime shift		Post-regime shift	
	π	$\pi(S)$	π	$\pi(S)$
σ	Aggressive	Ambiguous	Precautionary	Ambiguous
$\hat{P}$	No effect	Ambiguous	No effect	Ambiguous
γ	Ambiguous	Ambiguous	Precautionary	Ambiguous
δ	Ambiguous	Ambiguous	Precautionary	Ambiguous

Table 2: With resource scarcity: Effects of a potential regime shift threat on optimal fossil-fuel use for the four potential tipping elements $\{\sigma, \hat{P}, \gamma, \delta\}$.

Compared to an exogenous hazard rate, an endogenous hazard rate implies that the results are in all cases are ambiguous and will depend on the chosen parameter values and functional forms. In section G of appendix we provide examples for combinations of parameter values that can characterize the ambiguous results found in table 2 for this case. In the pre-shift environment the point is thus to show that depending on the magnitude of the shifts and the shape of the hazard rate function both precautionary and aggressive policy behavior may result, which reveals that the green paradox effect may actually dominate for some specific parameter combinations. Meanwhile, when the shift occurs, the jump in the control variable E may be either upward or downward.

The complete qualitative results of a potential regime shift on policy with both exogenous and endogenous hazard rate is outlined in table 2. In contrast to the section 5.1 we see that when resources are scarce the impact on fossil-fuel use in the pre-shift era is in general ambiguous. In particular, by proposition 7 we see that there is a dampening green paradox effect when resources are scarce which implies that the threat of a potential regime shift will not impact as strongly on fossil-fuel use at the outset as in the case when resources are abundant. Whether the damage effect or the Green Paradox effect dominates is in general ambiguous except for the case when probabilities are constant and the regime shift acts on the parameters σ or $\hat{P}$.

5.3 The role of uncertainty

As was seen in the previous sections the impact of uncertainty on optimal management with respect to a potential regime shift varies greatly depending on what one assumes with respect to resource availabilities, hazard rates and what tipping elements one considers. To make some more sense of how uncertainty impacts on optimal management we will look closer at an example when the hazard rate is constant and compare this to a deterministic case where the present value of marginal damages are equal to the expected present value of marginal damages under uncertainty. This will allow us to separate out the impact of uncertainty on optimal management.

Hence, consider a regime shift affecting for example γ or δ. We can then use a parallel deterministic case for comparison to gain some further understanding of the role of uncertainty in the model. Before the regime shift we will have $\Theta = 0$ and $\tilde{\Gamma}(\pi)$ will be a constant which we can compute given we know the probabilities and parameter values. We can thus compare the solution for this case where a potential regime shift is present to a deterministic case (without a potential regime shift) with a present value of marginal damages equal to $\bar{\Gamma}$ and where the parameters have been calibrated such that $\bar{\Gamma} = \tilde{\Gamma}(\pi)$. In this case the expected present value of the marginal damages before the regime shift has happened is the same as the present value of marginal damages in the deterministic case used for comparison. We can state the following proposition.

Proposition 10. Compare a case with a constant probability of a regime shift that affects γ or δ (but does not affect σ or $\hat{P}$) to a case without regime shift where the parameters are such that $\bar{\Gamma} = \tilde{\Gamma}(\pi)$. Assuming that we start from the same initial state variables, first-period fossil-fuel use will be higher in the case with a potential regime shift.

Proof. In both cases, first-period fossil-fuel use will be based on the same (expected) marginal damages. The other factor determining fossil-fuel use is the shadow value V_R of the resource stock R. In proposition 12 in appendix E, we prove that in the realizations where the regime shift has not happened yet, the shadow value grows faster than $\frac{1}{\beta}$ while it grows at the rate $\frac{1}{\beta}$ in the deterministic case. Assuming that the initial shadow value V_R is larger in the case with a regime shift compared to the deterministic case would imply that as long as the regime shift has not happened yet, the shadow value would be larger in all periods and consequently that fossil-fuel use would be lower in the case with a potential regime shift (conditional on that the regime shift has not happened yet) compared to the deterministic case. Since all fossil fuel should be used in all realizations this leads to a contradiction. We can therefore conclude that the initial shadow value V_R must be smaller in the case with a potential regime shift compared to the deterministic case and, consequently, that initial fossil-fuel use is higher in the case with a potential regime shift compared to the deterministic case. □

Figure 5.3 highlights the effects of uncertainty on optimal management of fossil-fuel use. When γ or δ are affected the qualitative results will be as is depicted in figure 5.3 i.e. optimal fossil-fuel use is initially higher under uncertainty compared to the deterministic case. This effect is related to the fact that the value of the resource will be potentially lower in the future due to the regime shift which creates an incentive to increase usage now. This result is likely related to the fact that the model assumes that consumers are risk averse. The opposite would likely be the case had this not been true.

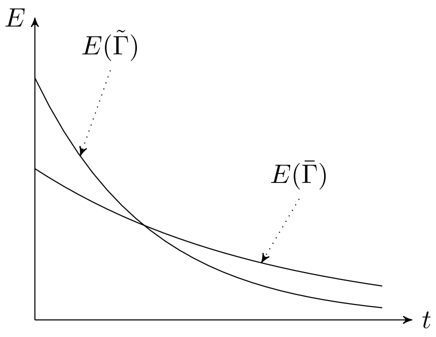

Figure 4: Comparison of optimal fossil-fuel use over under certainty $E(\bar{\Gamma})$ and uncertainty $E(\tilde{\Gamma})$.

6 Concluding remarks

In this paper we have analyzed how the threat of a potential regime shift affects the optimal time path of fossil-fuel use. We have found that the qualitative effects depend crucially on the set of modeling assumptions made. In particular we have considered whether initial fossil-fuel use is increased or decreased and whether fossil-fuel use increases or decreases when the shift happens. In both cases we find that the response of optimal fossil-fuel use could go either way depending on what tipping element we are considering, whether or not we think that finiteness of fossil-fuel resources is an important factor in determining optimal use and whether we consider the probability of the regime shift to be exogenous or endogenous.

Some general principles can, however, be distinguished. The total effect of introducing the potential for a regime shift into the model has three different parts that can be additively separated in the condition for optimal fossil-fuel use. Firstly, the marginal damages caused by emissions increase. Secondly, the probability of the shift happening increases with emissions. Both of these effects tend to decrease optimal fossil-fuel use. Thirdly, scarcity of fossil fuels imply that there is a shadow value of fossil-fuel resources. Increased damages tends decreases this shadow value leading to an opposing effect that tends to increase fossil-fuel use in all periods. The net effects will thus depend on assumptions made that affect the relative strengths of these effects.

If we assume that scarcity of fossil-fuel resources is not an issue(i.e., the shadow value of fossil-fuel resources is zero), there will be no opposing scarcity effect and the net effect will be to weakly decrease fossil-fuel use in all periods. Otherwise, the net effect will depend on the relative strength of the different effects. The effect of the introduction of a potential regime-shift is to make fossil-fuel use (weakly) less desirable in all periods. Optimal use will increase in periods where this effect is relatively weak and decrease in periods where this effect is relatively strong.

These results contrast with many other results that can be found in the literature on regime shifts and climate change which generally calls for precautionary measures due to the threat of a potential tipping point. In line with this literature our results also call for an increase in the optimal carbon tax when there is a threat of a potential regime shift. However, when the future value of the resource stock is also considered the net effect on optimal fossil-fuel use in many cases becomes ambiguous. Similar results to ours but for the case of natural resource management has also recently been found in (Ren and Polasky, 2013). Our paper is however the first to our knowledge that lifts the impact of resource constraints on optimal management under potential regime shifts. This suggests that further work is needed in order to better understand better what the qualitative effects involved in these settings.

A Solution to the post-shift problem

We will here go through the details of solving the post-shift problem described in the Bellman equation (7). Taking first-order conditions gives us

$$K' \quad : \quad U'(C) = \beta \hat{V}_K(K', R', S') \tag{20}$$

$$E \quad : \quad U'(C)\hat{D}(S)F_E(K,E) = \beta \left[\hat{V}_R(K', R', S') - \hat{\sigma} \hat{V}_S(K', R', S') \right]. \tag{21}$$

From the envelope condition we know that the partial derivatives of the value function with respect to the state variables will only include direct effects. Differentiating the maximized version of the right-hand side of (7) with respect to K we get

$$\hat{V}_K(K,R,S) = U'(C)\hat{D}(S)F_K(K,E). \tag{22}$$

Moving this forward one period and substituting in (20) we get the Euler equation

$$U'(C) = \beta U'(C')\hat{D}(S')F_K(K',E'). \tag{23}$$

Differentiating the the maximized version of the right-hand side of (7) with respect to R we get

$$\hat{V}_R(K,R,S) = \beta\hat{V}_R(K',R',S') \tag{24}$$

implying that the scarcity value of the resource increases at the rate $\frac{1}{\beta}$ between periods.

Differentiating the the maximized version of the right-hand side of (7) with respect to S we get

$$\hat{V}_S(K,R,S) = U'(C)\frac{D'(S)}{\hat{D}(S)}\hat{D}(S)F(K,E) + \beta(1-\hat{\delta})\hat{V}_S(K',R',S').$$

Substituting repeatedly for $\hat{V}$ in future time periods yields

$$\begin{aligned}\hat{V}_S(K,R,S) \quad = \quad & \sum_{n=0}^{N-1}\beta^n(1-\hat{\delta})^n U'(C^{(n)})\frac{\hat{D}'\left(S^{(n)}\right)}{\hat{D}\left(S^{(n)}\right)}\hat{D}\left(S^{(n)}\right)F\left(K^{(n)},E^{(n)}\right) \\ & +\beta^N(1-\hat{\delta})^N V_S\left(K^{(N)},R^{(N)},S^{(N)}\right).\end{aligned}$$

Assuming that $\hat{V}_S$ is bounded (follows from finiteness of the resource for "reasonable" $\hat{D}$) the last term goes to zero as $N \to \infty$ and we get

$$\hat{V}_S(K,R,S) = \sum_{n=0}^{\infty}\beta^n(1-\hat{\delta})^n U'(C^{(n)})\frac{\hat{D}'\left(S^{(n)}\right)}{\hat{D}\left(S^{(n)}\right)}\hat{D}\left(S^{(n)}\right)F\left(K^{(n)},E^{(n)}\right). \tag{25}$$

We could substitute (24) and (25) into (21) directly but we will instead start by utilizing some of our functional-form assumptions (3a)-(3d). With aggregate production given by (1) and the Cobb-Douglas production of (3c) we thus have that

$$\hat{D}(S)F_K(K,E) = \alpha\frac{Y}{K} \text{ and } \hat{D}(S)F_E(K,E) = \nu\frac{Y}{E}. \tag{26}$$

Furthermore, the exponential form of the damage function (3d) implies

$$\frac{\hat{D}'(S)}{\hat{D}(S)} = -\hat{\gamma}. \tag{27}$$

We can now rewrite the Euler equation (23) as

$$\frac{1}{C} = \beta\frac{1}{C'}\alpha\frac{Y'}{K'}.$$

This is fulfilled by the constant savings rate $\alpha\beta$

$$C = (1-\alpha\beta)Y \text{ and } K' = \alpha\beta Y. \tag{28}$$

We can also simplify the derivative of the maximized value function with respect to S by applying (27) and (8)

$$\begin{aligned}
\hat{V}_S(K,R,S) &= \sum_{n=0}^{\infty}\beta^n(1-\hat{\delta})^n\frac{\hat{D}'\left(S^{(n)}\right)}{\hat{D}\left(S^{(n)}\right)}\frac{Y^{(n)}}{C^{(n)}} = -\frac{\hat{\gamma}}{1-\alpha\beta}\sum_{n=0}^{\infty}\beta^n(1-\hat{\delta})^n \\
&= -\frac{\hat{\gamma}}{1-\alpha\beta}\frac{1}{1-\beta(1-\hat{\delta})} \equiv -\hat{\Gamma}.
\end{aligned}$$

Hence, we see that the marginal externality cost of emissions becomes constant in the post-shift environment.

Substituting the scarcity value of the resource (24) and the shadow value of the climate state (3) into the first-order condition with respect to E (21) we get

$$\begin{aligned}
U'(C)\hat{D}(S)F_E(K,E) &= \beta\left[\hat{V}_R(K',R',S') - \hat{\sigma}\hat{V}_S(K',R',S')\right] \\
&= \hat{V}_R(K,R,S) + \beta\hat{\sigma}\hat{\Gamma}.
\end{aligned}$$

Using the constant savings rate (8) and the marginal product of energy from (26), the left-hand side can be simplified to give

$$\frac{\nu}{E}\frac{1}{1-\alpha\beta} = \hat{V}_R(K,R,S) + \beta\hat{\sigma}\hat{\Gamma} \Rightarrow E = \frac{\nu}{(1-\alpha\beta)\left(\hat{V}_R(K,R,S) + \beta\hat{\sigma}\hat{\Gamma}\right)}.$$

Moving this forward and using (24) we have that

$$\begin{aligned} E^{(n)} &= \frac{\nu}{(1-\alpha\beta)\left(\hat{V}_R\left(K^{(n)},R^{(n)},S^{(n)}\right) + \beta\hat{\sigma}\hat{\Gamma}\right)} \\ &= \frac{\nu}{(1-\alpha\beta)\left(\beta^{-n}\hat{V}_R\left(K,R,S\right) + \beta\hat{\sigma}\hat{\Gamma}\right)}. \end{aligned} \tag{29}$$

If the constraint on total amount of available fossil-fuel does not bind (it will always bind when the resource is finite) we have that $V_R = 0$ and

$$E^{(n)} = \frac{\nu}{(1-\alpha\beta)\beta\hat{\sigma}\hat{\Gamma}}$$

for all n. In the following we will assume that the constraint does bind. We then have that

$$R = \sum_{n=0}^{\infty} E^{(n)} = \frac{\nu}{1-\alpha\beta}\sum_{n=0}^{\infty}\frac{1}{\beta^{-n}\hat{V}_R\left(K,R,S\right) + \beta\hat{\sigma}\hat{\Gamma}}. \tag{30}$$

This implicitly gives the value of $\hat{V}_R(K,R,S)$ as a function of the state variables. We can note that this expression does not contain state variables K and S implying that $\hat{V}_R(K,R,S)$ is actually independent of them. Without actually solving for $\hat{V}_R$, we can still determine how it depends on R and the regime specific parameters. The right-hand side is decreasing in $\hat{V}_R$, implying that if R increases $\hat{V}_R$ decreases capturing the decreased scarcity value of fossil fuel. If $\hat{\sigma}$ increases, $\hat{V}_R$ must decrease to maintain the right-hand side. This captures that a larger share of emissions ending up in the atmosphere decreases the value of burning fossil fuel, and consequently, the value of the resource stock. Similarly, an increase in $\hat{\Gamma}$, which measures the future damages caused by CO_2 in the atmosphere, gives a decrease in $\hat{V}_R$.

We can almost write down the value function explicitly. We start by rewriting $\hat{V}_K$ in (22) using the marginal product of capital from (26), logarithmic utility (3a) and the savings rule (8)

$$\hat{V}_K(K,R,S) = U'(C)\hat{D}(S)F_K(K,E) = \frac{\alpha}{1-\alpha\beta}\frac{1}{K}.$$

This depends only on K. From (30) we know that $\hat{V}_R$ only depends on R and from (3) we know that $\hat{V}_S$ is constant. This implies that $\hat{V}$ is seprarable and can be written as

$$\hat{V}(K,R,S) = \frac{\alpha}{1-\alpha\beta}\ln K + \hat{W}(R) - \hat{\Gamma}S,$$

where $\hat{W}(R)$ fulfills[10]

$$R = \frac{\nu}{1-\alpha\beta}\sum_{n=0}^{\infty}\frac{1}{\beta^{-n}\hat{W}'(R)+\beta\hat{\sigma}\hat{\Gamma}}$$

and could contain some constant term required to get the level of $\hat{V}$ right.

B Solution to the pre-shift problem

We will here go through the details of solving the pre-shift Bellman equation (13). The first-order condition with respect to K' is given in (14). Differentiating the maximized version of the right-hand side of (13) and using the envelope condition to get rid of the indirect effects we get

$$V_K(K,R,S) = U'(C)D(S)F_K(K,E) \Rightarrow V'_K = U'(C')D(S')F_K(K',E').$$

[10]To make this sole dependence on R clear we have thus defined $\hat{V}_R \equiv W'(R)$.

If the regime shift happens, this will not affect the next period state variables. Using (22) we can write the first-order condition with respect to K as

$$\begin{aligned} U'(C) =& \beta \left(1-\pi(S')\right) U'(C')D(S')F_K(K',E') \\ &+ \beta\pi(S')U'(\hat{C}')\hat{D}(S'+\hat{P})F_K(K',\hat{E}'). \end{aligned}$$

Using the assumptions of a logarithmic utility function (3a) and Cobb-Douglas production function (3c) we can simplify this to

$$\frac{K'}{C} = \alpha\beta \left[\left(1-\pi(S')\right) \frac{Y'}{C'} + \pi(S')\frac{\hat{Y}'}{\hat{C}'} \right].$$

We can see that the constant consumption/savings rate (8) still applies. The constant consumption/savings rate reduces complexity by reducing the dimensionality of the problem which is shown in the following proposition

Proposition 11. Given the assumptions (3a)-(3b) the value function can now be written in the following form

$$V(K,S,R) = \frac{\alpha}{1-\alpha\beta}\ln K + W(R,S)$$

Proof. To see this we simply proceed by guessing that the solution has the same form as we found in the post-regime shift section i.e. as in (15). Substituting (15) and the optimal consumption and saving rules (8) into the right hand side of the Bellman equation (13) this becomes

$$\begin{aligned} \ln((1-\alpha\beta)D(S)K^\alpha E^\nu) &+ \beta\left(1-\pi(S')\right)\left(\frac{\alpha}{1-\alpha\beta}\ln(\alpha\beta D(S)K^\alpha E^\nu) + W(R',S')\right) \\ &+ \beta\pi(S')\left(\frac{\alpha}{1-\alpha\beta}\ln(\alpha\beta D(S)K^\alpha E^\nu) + \hat{W}'(R,S)\right) \\ &= \mathbf{C} + \frac{\alpha}{1-\alpha\beta}\ln(K) + \frac{1}{1-\alpha\beta}\ln(D(S)E^\nu) + \\ &+ \beta\left[\left(1-\pi(S')\right)W(R',S') + \beta\pi(S')\hat{W}(R',S')\right] \end{aligned}$$

where $\hat{W}(R,S) \equiv \hat{W}(R) - \hat{\Gamma}S$ and $\mathbf{C} = \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}$ from (11). Comparing this expression with the left hand side of the Bellman equation we see that the term $\frac{\alpha}{1-\alpha\beta}\ln(K)$ cancels out which shows that the form in (15) satisfies the optimality condition. $\square$

The first-order condition with respect to E is given in (16). Differentiating the maximized value function with respect to R we get

$$V_R(K,R,S) = \beta\left[(1-\pi(S'))\,V_R' + \pi(S')\hat{V}_R'\right]. \tag{31}$$

This says that the expected shadow value of the resource stock should increase at the rate $\frac{1}{\beta}$.

Differentiating the maximized value function with respect to S we get

$$\begin{aligned} V_S(K,R,S) \quad &= \quad U'(C)D'(S)F(K,E) \\ &\quad +\beta(1-\delta)\left[(1-\pi(S'))\,V_S' + \pi(S')\hat{V}_S'\right] \\ &\quad +\beta(1-\delta)\pi'(S')\left[\hat{V}' - V'\right] \end{aligned} \tag{32}$$

We can simplify the first term using the exponential form of the damage function (3d), logarithmic utility (3a) and the savings rule (8)

$$U'(C)D'(S)F(K,E) = U'(C)\frac{D'(S)}{D(S)}D(S)F(K,E) = -\frac{\gamma}{1-\alpha\beta}.$$

From (3) we have that $\hat{V}_S^{(n)} = -\hat{\Gamma}$ for all n and we can substitute forwards in (32)

$$\begin{aligned} V_S(K,R,S) \quad &= \quad -\frac{\gamma}{1-\alpha\beta}\sum_{n=0}^{N-1}\beta^n(1-\delta)^n\left(1-\pi\left(S^{(n)}\right)\right)\Omega_1^n \\ &\quad -\hat{\Gamma}\sum_{n=1}^{N}\pi\left(S^{(n)}\right)\beta^n(1-\delta)^n\Omega_1^n \\ &\quad +\sum_{n=1}^{N}\beta^n(1-\delta)^n\pi'\left(S^{(n)}\right)\left(\hat{V}^{(n)} - V^{(n)}\right)\Omega_1^n \\ &\quad +\beta^N(1-\delta)^N V_S^{(N)}\Omega_1^{N+1}. \end{aligned}$$

As before, we assume that $V_S^{(N)}$ is bounded which implies that the last term goes to zero as $N \to \infty$. Applying this limit and moving forward one period we can write V_S' from (16) as

$$
\begin{aligned}
V_S' &= -\frac{\gamma}{1-\alpha\beta}\sum_{n=0}^{\infty}\beta^n(1-\delta)^n\left(1-\pi\left(S^{(n+1)}\right)\right)\Omega_2^{n+1} \\
&\quad -\hat{\Gamma}\sum_{n=1}^{\infty}\pi\left(S^{(n+1)}\right)\beta^n(1-\delta)^n\Omega_2^{n+1} \\
&\quad +\sum_{n=1}^{\infty}\beta^n(1-\delta)^n\pi'\left(S^{(n+1)}\right)\left(\hat{V}^{(n+1)}-V^{(n+1)}\right)\Omega_2^{n+1}.
\end{aligned}
$$

Returning to the first order condition with resect to E in (16), the left-hand side can be simplified to

$$
U'(C)D(S)F_E(K,E) = \frac{D(S)F_E(K,E)}{(1-\alpha\beta)D(S)F(K,E)} = \frac{\nu}{1-\alpha\beta}\frac{1}{E}.
$$

Substituting this, $\hat{V}_S' = -\hat{\Gamma}$ and (31) in (16) and using that

$$
(1-\pi(S'))\,\Omega_2^{n+1} = \Omega_1^{n+1}
$$

gives

$$
\begin{aligned}
\frac{\nu}{1-\alpha\beta}\frac{1}{E} &= V_R + \beta\sigma\sum_{n=0}^{\infty}\beta^n(1-\delta)^n\pi'\left(S^{(n+1)}\right)\left(V^{(n+1)}-\hat{V}^{(n+1)}\right)\Omega_1^{n+1} \\
&\quad +\beta\sigma\frac{\gamma}{1-\alpha\beta}\sum_{n=0}^{\infty}\beta^n(1-\delta)^n\left(1-\pi\left(S^{(n+1)}\right)\right)\Omega_1^{n+1} \\
&\quad +\beta\sigma\hat{\Gamma}\sum_{n=0}^{\infty}\pi\left(S^{(n+1)}\right)\beta^n(1-\delta)^n\Omega_1^{n+1}.
\end{aligned}
$$

Using definitions (18) and (19) we arrive at (17).

C Proof of proposition 2

We will here show how $\tilde{\Gamma}$ relates to Γ and $\hat{\Gamma}$.

We will start by computing a sum that we need in order to do this.

$$\begin{aligned}
& \frac{1}{1-\beta(1-\delta)} \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \pi\left(S^{(n+1)}\right) \Omega_1^{n+1} \\
= & \sum_{l=0}^{\infty} \beta^l (1-\delta)^l \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \pi\left(S^{(n+1)}\right) \Omega_1^{n+1} \\
= & \sum_{l=0}^{\infty} \beta^l (1-\delta)^l \left[\pi\left(S'\right) + \beta(1-\delta)\pi\left(S''\right)\Omega_1^2 + \beta^2(1-\delta)^2 \pi\left(S^{(3)}\right) + \ldots\right] \\
= & \pi\left(S'\right) + \beta(1-\delta)\left[\pi(S') + \pi\left(S''\right)\Omega_1^2\right] \\
& + \beta^2(1-\delta)^2 \left[\pi\left(S'\right) + \pi\left(S''\right)\Omega_1^2 + \pi\left(S^{(3)}\right)\Omega_1^3\right] + \ldots \\
= & \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \sum_{l=1}^{n+1} \pi\left(S^{(l)}\right) \Omega_1^l.
\end{aligned}$$

The second (inner) sum in the last expression is the probability that the regime shift happens in some period $1, \ldots, n+1$ and therefore it is also the complementary event to the event that there is no regime shift before period $n+2$ implying that

$$\sum_{l=1}^{n+1} \pi\left(S^{(l)}\right) \Omega_1^l = 1 - \Omega_1^{n+2}$$

and we have

$$\frac{1}{1-\beta(1-\delta)} \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \pi\left(S^{(n+1)}\right) \Omega_1^{n+1} = \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \left[1 - \Omega_1^{n+2}\right]. \quad (33)$$

We will now first show that $\tilde{\Gamma} \geq \Gamma$ and after that we will show that $\tilde{\Gamma} \leq \hat{\Gamma}$. For convenience we start by restating $\tilde{\Gamma}$ as defined in (19).

$$\begin{aligned}
\tilde{\Gamma}\left(\{S^{(l)}\}\right) \quad = \quad & \frac{\gamma}{1-\alpha\beta} \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \left(1 - \pi\left(S^{(n+1)}\right)\right) \Omega_1^{n+1} \\
& + \hat{\Gamma} \sum_{n=0}^{\infty} \pi\left(S^{(n+1)}\right) \beta^n (1-\delta)^n \Omega_1^{n+1} \qquad (34)
\end{aligned}$$

C.1 $\tilde{\Gamma} \geq \Gamma$

The second sum of (34) is

$$\begin{aligned}
& \hat{\Gamma} \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \pi \left(S^{(n+1)} \right) \Omega_1^{n+1} \\
= \quad & \frac{\hat{\gamma}}{1-\alpha\beta} \frac{1}{1-\beta(1-\hat{\delta})} \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \pi \left(S^{(n+1)} \right) \Omega_1^{n+1} \\
= \quad & \left\{ \begin{array}{ccc} \frac{1}{1-\beta(1-\hat{\delta})} & = & \frac{1}{1-\beta(1-\delta)} + \frac{1}{1-\beta(1-\hat{\delta})} - \frac{1}{1-\beta(1-\delta)} \\ & = & \frac{1}{1-\beta(1-\delta)} + \frac{\beta(\delta-\hat{\delta})}{(1-\beta(1-\delta))\left(1-\beta(1-\hat{\delta})\right)} = \end{array} \right\} \\
= \quad & \frac{\hat{\gamma}}{1-\alpha\beta} \frac{1}{1-\beta(1-\delta)} \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \pi \left(S^{(n+1)} \right) \Omega_1^{n+1} \\
& + \frac{\hat{\gamma}}{1-\alpha\beta} \frac{\beta(\delta-\hat{\delta}) \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \pi \left(S^{(n+1)} \right) \Omega_1^{n+1}}{(1-\beta(1-\delta))\left(1-\beta(1-\hat{\delta})\right)} = \{(33)\} = \\
= \quad & \frac{\hat{\gamma}}{1-\alpha\beta} \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \left[1 - \Omega_1^{n+2} \right] \\
& + \frac{\hat{\gamma}}{1-\alpha\beta} \frac{\beta(\delta-\hat{\delta}) \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \pi \left(S^{(n+1)} \right) \Omega_1^{n+1}}{(1-\beta(1-\delta))\left(1-\beta(1-\hat{\delta})\right)}.
\end{aligned}$$

We also have that the first sum of the right-hand side of (34) is

$$\frac{\gamma}{1-\alpha\beta} \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \left(1 - \pi \left(S^{(n+1)} \right) \right) \Omega_1^{n+1} = \frac{\gamma}{1-\alpha\beta} \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \Omega_1^{n+2}.$$

We can now compute $\tilde{\Gamma}$ as

$$\begin{aligned}
\tilde{\Gamma} \quad = \quad & \frac{\gamma}{1-\alpha\beta} \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \Omega_1^{n+2} + \frac{\hat{\gamma}}{1-\alpha\beta} \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \left[1 - \Omega_1^{n+2} \right] \\
& + \frac{\hat{\gamma}}{1-\alpha\beta} \frac{\beta(\delta-\hat{\delta}) \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \pi \left(S^{(n+1)} \right) \Omega_1^{n+1}}{(1-\beta(1-\delta))\left(1-\beta(1-\hat{\delta})\right)} \\
= \quad & \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \left(\frac{\gamma}{1-\alpha\beta} \Omega_1^{n+2} + \frac{\hat{\gamma}}{1-\alpha\beta} \left[1 - \Omega_1^{n+2} \right] \right) \\
& + \frac{\hat{\gamma}}{1-\alpha\beta} \frac{\beta(\delta-\hat{\delta}) \sum_{n=0}^{\infty} \beta^n (1-\delta)^n \pi \left(S^{(n+1)} \right) \Omega_1^{n+1}}{(1-\beta(1-\delta))\left(1-\beta(1-\hat{\delta})\right)}.
\end{aligned}$$

From this we can see that if $\hat{\gamma} \geq \gamma$ and $\hat{\delta} \leq \delta$ with at least one strict inequality, then $\tilde{\Gamma} > \Gamma$ since the first sum is larger than Γ and the second sum is positive.

C.2 $\tilde{\Gamma} \leq \hat{\Gamma}$

Starting from (34), $\tilde{\Gamma}$ can be rewritten as

$$\begin{aligned}
\tilde{\Gamma} &= \frac{\gamma}{1-\alpha\beta}\sum_{n=0}^{\infty}\beta^n\left[\left(1-\hat{\delta}\right)^n+(1-\delta)^n-\left(1-\hat{\delta}\right)^n\right]\left(1-\pi\left(S^{(n+1)}\right)\right)\Omega_1^{n+1} \\
&\quad +\hat{\Gamma}\sum_{n=0}^{\infty}\beta^n\left[\left(1-\hat{\delta}\right)^n+(1-\delta)^n-\left(1-\hat{\delta}\right)^n\right]\pi\left(S^{n+1}\right)\Omega_1^{n+1} \\
&= \frac{\gamma}{1-\alpha\beta}\sum_{n=0}^{\infty}\beta^n\left(1-\hat{\delta}\right)^n\left(1-\pi\left(S^{(n+1)}\right)\right)\Omega_1^{n+1} \\
&\quad +\frac{\hat{\gamma}}{1-\alpha\beta}\frac{1}{1-\beta\left(1-\hat{\delta}\right)}\sum_{n=0}^{\infty}\beta^n\left(1-\hat{\delta}\right)^n\pi\left(S^{(n+1)}\right)\Omega_1^{n+1} \\
&\quad +\frac{\gamma}{1-\alpha\beta}\sum_{n=0}^{\infty}\beta^n\left[(1-\delta)^n-\left(1-\hat{\delta}\right)^n\right]\left(1-\pi\left(S^{(n+1)}\right)\right)\Omega_1^{n+1} \\
&\quad +\hat{\Gamma}\sum_{n=0}^{\infty}\beta^n\left[(1-\delta)^n-\left(1-\hat{\delta}\right)^n\right]\pi\left(S^{(n+1)}\right)\Omega_1^{n+1}.
\end{aligned}$$

Using (33) with $\delta = \hat{\delta}$ and that $\left(1-\pi\left(S^{(n+1)}\right)\right)\Omega_1^{n+1} = \Omega_1^{n+2}$ we arrive at

$$\begin{aligned}
\tilde{\Gamma} &= \sum_{n=0}^{\infty}\beta^n\left(1-\hat{\delta}\right)^n\left[\Omega_1^{n+2}\frac{\gamma}{1-\alpha\beta}+\left(1-\Omega_1^{n+2}\right)\frac{\hat{\gamma}}{1-\alpha\beta}\right] \\
&\quad +\sum_{n=0}^{\infty}\beta^n\left[(1-\delta)^n-\left(1-\hat{\delta}\right)^n\right]\left[\Omega_1^{n+2}\frac{\gamma}{1-\alpha\beta}+\pi\left(S^{(n+1)}\right)\Omega_1^{n+1}\hat{\Gamma}\right].
\end{aligned}$$

We can see that $\tilde{\Gamma} \leq \hat{\Gamma}$ since the first sum is smaller than $\hat{\Gamma}$ and the second sum is negative.

D Proof of proposition 3

Assume that the regime shift has happened. The optimization problem from then on is then deterministic. The solution is given by the solution to the post-shift problem and will prescribe using the consumption/savings rule (8) and some sequence of fossil-fuel use $\{\hat{E}^{(l)}\}$. Assuming instead that the regime shift has not happened yet. We could then still use the same consumption/savings rule and the same sequence of fossil-fuel use until the shift happens and then follow the optimal post-shift solution from then on. We will now show that this would generate more utility. Since this would typically not be optimal (since decisions prior to the shift are not chosen optimally), the optimal solution would generate even more utility. We will start by comparing the utility if the shift has happened in the current period to the case where it happens in the next period. Let the values in the two cases be V_1 and V_2 respectively. The computations use the consumption/savings rule (8) and the explicit expression for the post-shift value function (11). The value when the shift happened in the current period is

$$\begin{aligned} V_1 &= \ln\left((1-\alpha\beta)e^{-\hat{\gamma}\left(S+\hat{P}\right)}K^{\alpha}E^{\nu}\right) \\ &\quad +\beta\left[\frac{\alpha}{1-\alpha\beta}\ln\left((1-\alpha\beta)e^{-\hat{\gamma}\left(S+\hat{P}\right)}K^{\alpha}E^{\nu}\right)\right] \\ &\quad +\beta\left[\hat{W}(R-E)-\hat{\Gamma}\left(\hat{\sigma}E+\left(1-\hat{\delta}\right)\left(S+\hat{P}\right)\right)\right] \\ &= \frac{1}{1-\alpha\beta}\ln\left(K^{\alpha}E^{\nu}\right)-\frac{\hat{\gamma}}{1-\alpha\beta}(S+\hat{P})+\beta\hat{W}(R-E) \\ &\quad -\beta\hat{\Gamma}\left(\hat{\sigma}E+(1-\hat{\delta})(S+\hat{P})\right)+\mathbf{C}, \end{aligned}$$

where

$$\mathbf{C}=\frac{(1-\alpha\beta)\ln(1-\alpha\beta)+\alpha\beta\ln(\alpha\beta)}{1-\alpha\beta}.$$

The value where the shift happens in the next period is

$$\begin{aligned} V_2 &= \ln\left((1-\alpha\beta)e^{-\gamma S}K^{\alpha}E^{\nu}\right) \\ &\quad +\beta\left[\frac{\alpha}{1-\alpha\beta}\ln\left((1-\alpha\beta)e^{-\gamma S}K^{\alpha}E^{\nu}\right)\right] \\ &\quad +\beta\left[\hat{W}(R-E)-\hat{\Gamma}\left(\sigma E+(1-\delta)\,S+\hat{P}\right)\right] \\ &= \frac{1}{1-\alpha\beta}\ln\left(K^{\alpha}E^{\nu}\right)-\frac{\gamma}{1-\alpha\beta}S+\beta\hat{W}(R-E) \\ &\quad -\beta\hat{\Gamma}\left(\sigma E+(1-\delta)S+\hat{P}\right)+\mathbf{C}. \end{aligned}$$

The difference is

$$
\begin{aligned}
V_2 - V_1 &= \frac{1}{1-\alpha\beta}\left[-\gamma S + \hat{\gamma}(S+\hat{P})\right] - \beta\hat{\Gamma}\left[\sigma E + (1-\delta)S + \hat{P}\right] \\
&\quad + \beta\hat{\Gamma}\left[\hat{\sigma}E + (1-\hat{\delta})(S+\hat{P})\right] \\
&= \beta\hat{\Gamma}\left[(\hat{\sigma}-\sigma)E + \left(\delta-\hat{\delta}\right)S\right] + \frac{\hat{\gamma}-\gamma}{1-\alpha\beta}S + \left[\frac{\hat{\gamma}}{1-\alpha\beta} - \beta\hat{\delta}\hat{\Gamma}\right]\hat{P} \\
&= \beta\hat{\Gamma}\left[(\hat{\sigma}-\sigma)E + \left(\delta-\hat{\delta}\right)S\right] + \frac{\hat{\gamma}-\gamma}{1-\alpha\beta}S + \frac{\hat{\gamma}}{1-\alpha\beta}\frac{1-\beta}{1-\beta(1-\hat{\delta})}.
\end{aligned}
$$

This is positive implying that the value is higher if the shift happens in the next period compared to if it happened in the current period. Using the same logic, the value is higher the further into the future it happens when following the specified decision rules. This implies that it is always possible to achieve a higher value if the shift happens in the future compared to if it happened in the current period (when the state variables are otherwise the same as assumed in the proposition).

E Proposition 12

Proposition 12. Assume that a regime shift happens in period n, then by equation (24) we know that $\hat{V}_R$ grows at a rate equal to $\frac{1}{\beta}$ for all $s > n$. Assuming now that the regime shift does not happen in period n, then given a constant probability π of a regime shift, the pre-shift shadow value of the resource V_R must grow at a rate strictly larger than the growth rate of $\hat{V}_R$ for all periods $s > n$, as long as a regime-shift does not occur.

Proof. We can start by noting that after the regime shift, or in the case where the regime shift only causes a burst of methane (captured by $\hat{P}$) there are no changes in the marginal damages caused by emissions and V_R will grow exactly at the rate $\frac{1}{\beta}$. The remaining case is the case where the regime shift has not happened yet and where, when it happens, it will cause an increase in $\sigma\Gamma$. We will prove the proposition for this case by assuming that V_R grows at a rate smaller than $\frac{1}{\beta}$ and show that this leads to an inconsistency when comparing the realization where the regime happens in the next period to the realization where it never happens. To do this we need some new notation. Above we let $X^{(n)}$ denote values n period into the future. For pre regime shift variables it was conditional on the regime shift not having happened and for post-shift variables it was conditional on that the regime shift happened in the current period. We will now need to consider post regime shift variables where the regime shift happened more periods ago. We will denote this by $\hat{X}^{(n_1,n_2)}$ where n_1 refers to the period when the regime shift happened and where $n_2 \geq n_1$ refers to the period that the variable value refers to. For instance, $\hat{V}_R^{(n)}$ would in this notation be equal to $\hat{V}_R^{(n,n)}$.

We will now show that the assumption $V_R' < \frac{1}{\beta} V_R$ leads to a contradiction implying that we must have that $V_R' \geq \frac{1}{\beta} V_R$. We thus assume

$$V_R' < \frac{1}{\beta} V_R. \tag{35}$$

Combining this assumption with $V_R = \beta \left[\pi \hat{V}_R{}' + (1-\pi) V_R' \right]$ the implication is that $\hat{V}_R' > \frac{1}{\beta} V_R$. We also know that after the regime shift, $\hat{V}_R$ grows at the rate $\frac{1}{\beta}$ giving us

$$(35) \Rightarrow \hat{V}_R{}^{(1,n)} = \frac{1}{\beta^{n-1}} \hat{V}_R' > \frac{1}{\beta^n} V_R \text{ for all } n \geq 1. \tag{36}$$

We now turn to consider the implications for $V_R^{(n)}$ of assuming (35). Combining (35) and (36) implies that

$$V_R' < \hat{V}_R'. \tag{37}$$

Since $\sigma\tilde{\Gamma} < \hat{\sigma}\tilde{\Gamma}$ (where $\tilde{\Gamma}$ is given by (19) with constant probability π), this implies that $E' > \hat{E}'$ and since $R' = \hat{R}'$ we get $\hat{R}^{(2,2)} < \hat{R}^{(1,2)}$. The last inequality implies that $\hat{V}_R^{(2,2)} > \hat{V}_R^{(1,2)} = \frac{1}{\beta} \hat{V}_R^{(1,1)}$. Combining this last inequality with (37) we get

$$\hat{V}_R^{(2,2)} > \frac{1}{\beta} V_R'. \tag{38}$$

Since we know that

$$V_R' = \beta \left[\pi \hat{V}_R^{(2,2)} + (1-\pi) V_R'' \right]$$

(38) implies that $V_R'' < \frac{1}{\beta} V_R'$. The conclusion is that if V_R grows at a rate smaller than $\frac{1}{\beta}$ between two periods when the regime shift does not happen, it will continue to grow slower than $\frac{1}{\beta}$ as long as the regime shift does not happen. Iterating on this implication we can see that

$$(35) \Rightarrow V_R^{(n)} < \frac{1}{\beta^n} V_R \text{ for all } n \geq 1. \tag{39}$$

Since (current period) E is independent of when the regime shift happens, we have $R' = \hat{R}'$. The total resource constraint will bind for all realisations implying that we must have

$$\sum_{n=1}^{\infty} E^{(n)} = \sum_{n=1}^{\infty} \hat{E}^{(n,1)}.$$

However, combining (36), (39) and $\sigma\tilde{\Gamma} < \hat{\sigma}\hat{\Gamma}$ we have

$$\sum_{n=1}^{\infty} E^{(n)} = \frac{\nu}{1-\alpha\beta} \sum_{n=1}^{\infty} \frac{1}{V_R^{(n)} + \beta\sigma\tilde{\Gamma}} > \sum_{n=1}^{\infty} \hat{E}^{(1,n)} = \frac{\nu}{1-\alpha\beta} \sum_{n=1}^{\infty} \frac{1}{\hat{V}_R^{(1,n)} + \beta\hat{\sigma}\hat{\Gamma}}.$$

The conclusion is that assuming (35) leads to a contradiction. The assumption (35) must thus be false. This leads us to conclude that we must have

$$V_R' \geq \frac{1}{\beta} V_R.$$

□

F Proof of proposition 7

Let variables for the situation when the regime shift does not cause any parameter changes be denoted by variables without tildes and the variables for the case when the regime shift does cause changes by tildes. Solving both problems will give sequences

$$\left\{E^{(n)}, R^{(n)}, V_R^{(n)}, \hat{V}_R^{(n)}, \Gamma^{(n)}, \pi^{(n)}\right\} \text{ and } \left\{\tilde{E}^{(n)}, \tilde{R}^{(n)}, \tilde{V}_R^{(n)}, \tilde{\hat{V}}_R^{(n)}, \tilde{\Gamma}^{(n)}, \tilde{\Theta}^{(n)}\tilde{\pi}^{(n)}\right\}$$

where the sequences are conditional upon that the regime shift has not happened yet (or in the case of $\hat{V}_R^{(n)}$ and $\tilde{\hat{V}}_R^{(n)}$ that it just happened). In both cases, fossil-fuel use will be given by

$$E^{(n)} = \frac{\nu}{1 - \alpha\beta} \frac{1}{V_R^{(n)} + \Theta^{(n)} + \beta\sigma^{(n)}\Gamma^{(n)}}$$

or the corresponding expression with tildes on all variables. In both cases we also have

$$V_R^{(n)} = \beta\left[\left(1 - \pi^{(n+1)}\right)V_R^{(n+1)} + \pi^{(n+1)}\hat{V}_R^{(n+1)}\right] \tag{40}$$

and the corresponding expression with tildes. Note that in the case where the regime shift does not cause any changes, the probabilities do not matter since the values of the variables with or without hats vill be the same. From proposition 1 we know that

$$\hat{V}_R \geq \tilde{\hat{V}}_R \text{ for given } R \text{ and that both of them are decreasing in } R. \tag{41}$$

and the inequality will be strict if at least one of $\hat{\delta} \leq \delta$, $\hat{\gamma} \geq \gamma$ and $\hat{\sigma} \geq \sigma$ are strict.

We will now show that assuming that $\tilde{V}_R \geq V_R$ (for a given initial R) leads to a contradiction. We can start by noting that for all n, $\tilde{\Theta}^{(n)} \geq \Theta^{(n)} = 0$ and $\tilde{\sigma}^{(n)}\tilde{\Gamma}^{(n)} \geq \sigma^{(n)}\Gamma^{(n)}$. We must also have that

$$R = \sum_{n=0}^{\infty} E^{(n)} = \sum_{n=0}^{\infty} \tilde{E}^{(n)}$$

since all fossil-fuel will be used in all realizations including the ones where the regime shift happens arbitrarily far into the future. Assume now that $\tilde{V}_R \geq V_R$. This implies that $\tilde{E} \leq E$ and consequently that $\tilde{R}' \geq R'$. If $\pi'(S) > 0$ for all S then $\tilde{\Theta} > 0$ and we have that $\tilde{E} < E$. If at least one of $\hat{\delta} \leq \delta$, $\hat{\gamma} \geq \gamma$ and $\hat{\sigma} \geq \sigma$ are strict then $\hat{V}_R' > \tilde{\hat{V}}_R'$. Combined, this implies that under the assumptions made, $\tilde{\hat{V}}_R' < \hat{V}_R'$. Combining $\tilde{V}_R \geq V_R$ and $\tilde{\hat{V}}_R' < \hat{V}_R'$ and using (40) we get $\tilde{V}_R' > V_R'$. This in turn implies that $\tilde{E}' < E'$ and $\tilde{R}'' > R''$. Using (41) this implies that $\tilde{\hat{V}}_R'' < \hat{V}_R''$. Using (40), $\tilde{V}_R' > V_R'$ and $\tilde{\hat{V}}_R'' < \hat{V}_R''$ implies that $\tilde{V}_R'' > V_R''$. This in turn implies that $\tilde{E}'' < E''$. Going on we can show that assuming $\tilde{V}_R \geq V_R$ implies that

$$\sum_{n=0}^{\infty} \tilde{E}^{(n)} < \sum_{n=0}^{\infty} E^{(n)}$$

since this inequality holds for each term in the sums. This contradicts that both sums should be equal to R and consequently proves that $V_R \geq \tilde{V}_R$ can not hold.

G Numerical simulations

This section outlines the details of the numerical simulations. We begin by specifying a hazard rate function.

$$\pi(S) = \begin{cases} \frac{\left(\frac{S}{\bar{S}-S}\right)^{\vartheta}}{1+\left(\frac{S}{\bar{S}-S}\right)^{\vartheta}} & \text{if } S < \bar{S} \\ 1 & \text{if } S \geq \bar{S} \end{cases} \tag{42}$$

where $\vartheta = 20$ will be baseline estimates with one exception, while $\bar{S}$ will mostly vary in the various simulations. This gives us a rather sharp hazard rate function as can be seen in the graph below.

The other baseline parameter estimates are provided in table 3. Most of them are fairly standard. We use a yearly discount rate of 3% so that the choice of utility function and discount rate replicates that of Nordhaus and Boyer (2000). The value of the income shares α and ν are taken from Golosov et al. (2014). The amount of emissions immediately discharged into the atmosphere σ and the long-run depreciation rate of carbon dioxide is set fairly arbitrary to 0.5 and 0.05 respectively, they are however close in magnitude to the values

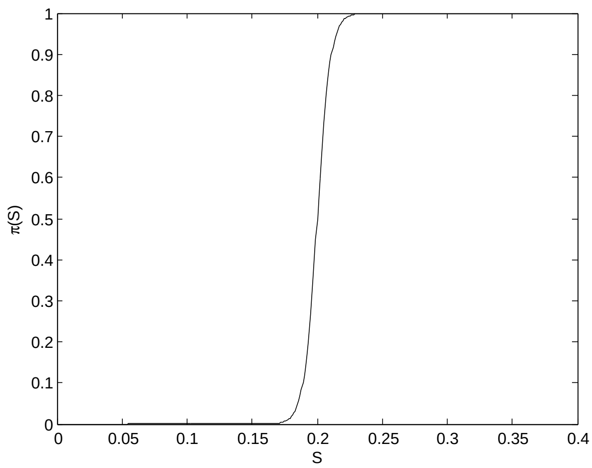

Figure 5: Baseline hazard rate function for $\bar{S} = 0.4$

used in Nordhaus (1994). The damage parameter γ is set based on the estimate in Golosov et al. (2014). However, since we throughout the simulations have normalized the initial stock of fossil-fuel reserves to unity we have also adjusted the damage parameter accordingly.[11] Finally, with respect to the hazard rate function these are quite arbitrary. As we have stressed several times already, the point here is not be quantitatively precise but rather to reveal qualitative behavior.

β	α	ν	δ	σ	γ
0.97^{10}	0.3	0.03	$1 - 0.95^{10}$	0.5	0.0212

Table 3: Baseline parameter estimates.

We are now interested in numerically showing that ambiguous cases may arise w.r.t whether optimal fossil-fuel use policy implies more or less extraction in when a potential regime shift is present as compared to the case when no such risk exists. Also we wish to show that optimal policy may also be ambiguous (i.e. shift either up or down as the policy adjusts to the post-shift environment) when/if a shift finally occurs. The tables below describes combinations of parameters that together with the baseline parameters can be used to numerically characterize/prove the ambiguity of optimal policy for the cases that we have not been able to characterize analytically.

Table 4 shows parameter combinations that makes optimal policy outcome ambiguous and dependent on the amount of resources left. Along the extraction path the extraction amount may shift either up or down once the shift has occured. This behavior is depicted in figure 2. Table 5 resolves the right hand graph in figure 3 i.e. when resources are scarce and the hazard rate is constant. From table 5 we see what combination of parameter values that gives precautionary and aggressive pre-shift behavior as a result of a potential regime shift being present. Table 6 shows which parameter combinations that result in aggressive and which result in precautionary pre-shift behavior when the hazard

[11]Our estimate of $\gamma = 0.0212$ is found by multiplying the Golosov et al. (2014) estimate by 400 which implies that damages from emissions will be comparitive.

rate is endogenous and resources are scarce. Finally, table 7 reveals the same parameter combinations also makes optimal policy outcome ambiguous and dependent on the amount of resources left implying that along the extraction path, the extraction amount may shift either up or down when the shift occurs. The Matlab code for solving these problems can be retrieved from the authors upon request.

Post-regime shift - endogenous probability $\pi(S)$ - No scarcity	
$\bar{S} = 4$, $\hat{\sigma} = 0.7$	Ambiguous
$\bar{S} = 4$, $\hat{\gamma} = 0.0312$	Ambiguous
$\bar{S} = 4$, $\hat{\delta} = 1 - 0.98^{10}$	Ambiguous

Table 4: Depending on the amount of resources left, optimal policy will imply a shift in E either up or down as the policy adjusts to the post-shift environment. The policy outcome is thus ambiguous.

Pre-regime shift - exogenous probability (π) - with scarcity	
$\pi = 0.05$, $\hat{\gamma} = 0.0312$	Precautionary
$\pi = 0.05$, $\hat{\delta} = 1 - 0.98^{10}$	Precautionary
$\pi = 0.001$, $\gamma = 4$, $\hat{\gamma} = 0.0312$	Aggressive
$\pi = 0.001$, $\delta = 1 - 0.5^{10}$, $\hat{\delta} = 1 - 0.95^{10}$	Aggressive

Table 5: Policy varies depending on the parameter values for the pre-regime shift, exogenous probability π and resource scarce case. In sum, the pre-shift policy is thus ambiguous.

Pre-regime shift - endogenous probability $\pi(S)$ - with scarcity	
$\bar{S} = 0.4$, $\hat{\sigma} = 0.7$	Aggressive
$\bar{S} = 0.4$, $\hat{P} = 0.1$	Aggressive
$\bar{S} = 0.4$, $\hat{\gamma} = 0.0312$	Aggressive
$\bar{S} = 0.4$, $\hat{\delta} = 1 - 0.98^{10}$	Aggressive
$\bar{S} = 0.3$, $\hat{\sigma} = 0.7$	Precautionary
$\bar{S} = 0.3$, $\hat{P} = 0.1$	Precautionary
$\bar{S} = 0.3$, $\hat{\gamma} = 0.0312$	Precautionary
$\bar{S} = 0.3$, $\hat{\delta} = 1 - 0.98^{10}$	Precautionary

Table 6: Policy varies depending on the parameter values of the hazard function for the pre-regime shift, endogenous probability $\pi(S)$ and resource scarce case. In sum, the pre-shift policy is thus ambiguous.

Post-regime shift - endogenous probability $\pi(S)$ - with scarcity	
$\bar{S} = 0.4$, $\hat{\sigma} = 0.7$	Ambiguous
$\bar{S} = 0.4$, $\hat{P} = 0.1$	Ambiguous
$\bar{S} = 0.4$, $\hat{\gamma} = 0.0312$	Ambiguous
$\bar{S} = 0.4$, $\hat{\delta} = 1 - 0.98^{10}$	Ambiguous

Table 7: Depending on the amount of resources left, optimal policy will imply a shift in E either up or down as the policy adjusts to the post-shift environment. The policy outcome is thus ambiguous.

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