FEEDBACK EQUILIBRIA FOR A CLASS OF NON-LINEAR DIFFERENTIAL GAMES IN RESOURCE ECONOMICS

By Anastasios Xepapadeas[1], Aart de Zeeuw[2] and Karl-Göran Mäler[3]

Abstract

The purpose of this paper is to develop an algorithm to solve a non-linear-quadratic differential game and to explicitly determine the (non-linear) feedback equilibrium strategies. In particular we consider a class of non-linear differential games that is often encountered in problems of environmental and resource management. At the first stage we analyse, as a benchmark, the cooperative outcome and the open-loop equilibria of the non-linear differential game. At the second stage we construct a procedure to determine the feedback equilibrium strategies numerically. The procedure makes use of the sufficient conditions for optimality in the Hamiltonian representation of the problem. Finally, an application to the shallow-lake non-linear differential game is given.

Keywords: Non-linear differential games, feedback strategies, multiple equilibria, shallow lakes.

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[1] Department of Economics, University of Crete, University Campus, 74 100, Rethymno , Greece, xepapad@econ.soc.uoc.gr

[2] Department of Economics and CentER, Tilburg University, A.J.deZeeuw@kub.nl

[3] The Beijer International Institute of Ecological Economics, The Royal Swedish Academy of Sciences, karl@beijer.kva.se

1. Introduction

Differential games have been extensively used during the recent decades to analyse economic problems in areas such as industrial organization, environmental and resource economics or macroeconomic policy.[4] In the differential game formalism two main solution concepts are the most often used: the open-loop Nash equilibrium (OLNE) concept where controls depend on the initial state of the system under investigation and time, and the feedback Nash equilibrium (FBNE) concept where attention is restricted to Markov perfect strategies with controls depending on the current state of the system.

As it is well known, the OLNE with an infinite period of commitment is weakly time-consistent but not strongly time-consistent (Basar, 1989). Therefore it does not possess the Markov perfect property and is not robust against unexpected changes in the state of the system. On the other hand, the FBNE is Markov perfect by construction and thus a more satisfactory solution concept, but solutions are usually very difficult to derive. Explicit solutions can be obtained by using a dynamic programming formulation for the so-called linear-quadratic differential game where the objective function is quadratic and the dynamics are linear. In this case a quadratic value function exists and this allows for an explicit solution of the problem.[5] When we depart from the linear-quadratic formulation, an explicit solution is not possible except for some cases where due to the specific structure of the game the feedback problem can be reduced to an open-loop problem (Fershtman, 1987).

Recent advances in environmental and resource economics emphasize the need for a realistic

[4]starting with among others Hoel (1978), Levhari and Mirman (1980), Reinganum and Stokey (1985) and Fershtman and Kamien (1987).

[5]The solution involves solving the matrix Riccati equation which determines endogenously a linear feedback (Markov-perfect) equilibrium strategy (e.g. Sargent, 1987).

representation of the natural system in unified economic-ecological models. Realistic modeling of natural systems in most cases indicates that the use of linear dynamics to model natural processes might not be a good approximation and that a non-linear formulation might be more appropriate. Non-linearities in the transition equations associated with the evolution of natural systems relate mainly to the existence of non-linear feedbacks which are physical processes that further impact an initial change of the system under investigation. Feedbacks could be positive if the impact is such that the initial perturbation is enhanced, or negative if the initial perturbation is reduced. For example in the study of climate change a positive feedback exists when an increase in temperature, say due to increased accumulation of greenhouse gases, increases evaporation from the oceans, which brings more water vapor into the atmosphere and finally enhances greenhouse effects.[6] In the analysis of shallow lake eutrophication, positive feedbacks are related to the release of phosphorus that has been slowly accumulated in sediments and submerged vegetation. Ignoring these non-linearities might obscure very important characteristics that we observe in reality such as bifurcations of the natural system to alternative equilibrium states, irreversibilities or hysteresis. It is important to note that the design of policies without taking into account the impact of non-linearities might lead to erroneous results and non-desirable states of the ecosystem.

If, however, non-linear feedbacks are introduced into the transition equations of differential game models for environmental and resource management, the often used linear-quadratic structure of the game is lost. This implies that the quadratic form of the value function cannot be used to determine the feedback equilibrium strategies. Thus the attempt to make the natural system more realistic complicates the use of the FBNE concept, since the non-linear structure

[6] Hansen et al. (1984).

of the differential game does not allow the standard determination of the feedback equilibrium strategies. We are not aware of many successful attempts to solve a non-linear-quadratic differential game and to explicitly determine the (non-linear) equilibrium strategies. An exception is the (stochastic) algorithm by Pakes and McGuire (1994, 2001) that computes Markov perfect equilibria to describe the dynamics of an industry with differentiated products. However, the algorithm presented in our paper is simpler and less computationally demanding by making use of two aspects. First, the specific structure of the type of problems we consider allows a transformation of the dynamic programming equation, so that we can iterate on the steady state of the FBNE instead of the space of equilibrium strategies and value functions. Second, considering the Hamiltonian system and the sufficiency conditions for optimality allows us to limit the area of search, so that we only need a local analysis. The purpose of this paper is to present and apply this algorithm. In particular we consider a class of non-linear differential games which are often encountered in problems of environmental and resource management. In these games many agents influence a common-pool resource the evolution of which is characterized by non-linear dynamics. For example, we may have individual countries emitting greenhouse gases and contributing to global warming, or farmers increasing phosphorus loadings to a shallow lake and contributing to its eutrophication. In both cases the realistic representation of the natural system requires the introduction of non-linear feedbacks in the system's transition equation.

At the first stage we analyse the cooperative solution and the OLNE solution of the underlying differential game. These solutions can be regarded as benchmark cases that expose the existence of multiple equilibria and of "good" and "bad" basins of attraction.

At the second stage we analyse the FBNE of the non-linear differential game. Using a transformation of the dynamic programming equation and insights from the Hamiltonian representation of the differential game, we are able to construct a procedure that determines the feedback equilibrium strategies numerically. Since the non-linearity of the problem induces multiple equilibria, our procedure determines the feedback equilibrium strategies locally in the basin of attraction of the corresponding open-loop Nash equilibrium. Thus our result also indicates that feedback equilibrium strategies may differ depending on the relevant basin of attraction.[7] We also show that when the sufficient conditions for optimality in the Hamiltonian representation of the differential game are satisfied, the FBNE steady state is "worse" than the OLNE steady state and thus worse than the cooperative one. This is a result contrary to the one obtained by Dockner and Long (1993) where it is stated that non-linear feedback strategies for a linear-quadratic differential game can reproduce the cooperative steady state. Our interpretation of this discrepancy is that in the Dockner-Long game the non-linear strategy is not a priori restricted so that the sufficient conditions of the Hamiltonian representation of the differential game are satisfied.

2. Cooperative Solution and Open-Loop Nash Equilibria

We consider a set of n agents, which could be countries or communities, that undertake a certain action $a_i \in \mathcal{A} \subset \mathcal{R}_+$, $i = 1, ..., n$, with which they affect the state of a natural system that is shared by all the agents. The action could be, for example, phosphorus loadings into a lake due to agricultural activities or emissions of greenhouse gases due to industrial activities. The action a_i undertaken at time t generates benefits according to a strictly increasing and concave utility function $U(a_i)$, which is assumed to be the same for all agents: $U'(a_i) > 0$, $U''(a_i) < 0$,

[7] This result is in agreement to a similar one obtained by Santos (2002) in the context of competitive-market economies with taxes and externalities.

$\lim_{a_i \to 0} U'(a_i) = +\infty$.

The evolution of pollutant in the natural system is described by the non-linear transition equation

$$\dot{x} = \sum_{i=1}^{n} a_i - bx + f(x), \; x(0) = x^0, \; x \in X \subset \mathcal{R}_+ \tag{2.1}$$

The state variable x could be interpreted, for example, as accumulated phosphorus in a lake or accumulated greenhouse gases. In (2.1) the function $f(x)$ is an increasing non-linear function of the state variable x reflecting the presence of non-linear feedbacks. We assume that $f(0) = 0$, $f'(0) = 0$, $\lim_{x \to \infty} f'(0) = 0$ with a unique $x_m = \text{argmax}\, f'(x)$ so that $f(x)$ is a convex-concave function, that is $f''(x) > 0$ for $x < x_m$ and $f''(x) < 0$ for $x > x_m$.[8] The accumulation of the state variable causes environmental damage (or equivalently reduces the flow of useful services generated by the natural system) according to a strictly increasing and convex damage function $D(x)$, which is also assumed to be the same for all agents: $D(0) = 0$, $D'(x) > 0$, $D''(x) > 0$. Thus the flow of net benefits accruing to each agent at any instant of time is given by

$$U(a_i(t)) - D(x(t))$$

Each agent is choosing a strategy a_i to maximize the present value of net benefits over an infinite time horizon, or

$$\max_{\{a_i(t)\}} J^i = \int_0^\infty e^{-\rho t} \left[U(a_i(t)) - D(x(t)) \right] dt, \; i = 1, ..., n \tag{2.2}$$

subject to (2.1), where $\rho > 0$ is a discount rate, common for all agents.

[8] For a full exposition of these assumptions and their implications see Brock and Starrett (2000).

A cooperative solution to the problem (2.2) is to choose the time paths $\{a_i(t)\}$ to maximize the sum of individual net benefits or,

$$\max_{\{a_1(t),...a_n(t)\}} \sum_{i=1}^{n} J^i = \int_0^\infty e^{-\rho t} \left[\sum_{i=1}^{n} U(a_i(t)) - nD(x(t)) \right] dt \tag{2.3}$$

subject to (2.1). The current-value Hamiltonian for problem (2.3) is defined as

$$\mathcal{H} = \sum_{i=1}^{n} U(a_i) - nD(x) + \lambda [a - bx + f(x)], \; a = \sum_{i=1}^{n} a_i$$

Pontryagin's maximum principle implies the following necessary conditions:[9]

$$U'(a_i) = -\lambda, \; i = 1, ..., n \tag{2.4}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial \mathcal{H}}{\partial x} = \left(\rho + b - f^{'}(x)\right)\lambda + nD^{'}(x) \tag{2.5}$$

$$\dot{x} = a - bx + f(x), \; x(0) = x^0 \tag{2.6}$$

along with the transversality conditions at infinity

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t) = 0, \lim_{t\to\infty} e^{-\rho t}\lambda(t)\, x(t) = 0$$

Solving (2.4) for a_i we obtain $a_i = U'^{-1}(-\lambda)$ for all i. Substituting this solution into (2.1) we obtain for the state equation:

$$\dot{x} = nU'^{-1}(-\lambda) - bx + f(x) \tag{2.7}$$

[9] The sufficient conditions are examined in the next section.

Equations (2.7) and (2.5) form the Modified Hamiltonian Dynamic System (MHDS) for the optimal control problem associated with the cooperative solution. A steady state for this system is defined as $\left(\bar{x}, \bar{\lambda}\right) : \dot{x} = 0, \dot{\lambda} = 0$ with $\bar{a_i} = U'^{-1}(-\bar{\lambda})$.

A MHDS for the cooperative solution can be equivalently defined in the state-control space. Differentiating (2.4) with respect to time we obtain

$$U^{''}\left(a_i\right)\dot{a}_i = -\dot{\lambda}$$

Substituting this into (2.5) we obtain

$$\dot{a}_i = \frac{1}{U''\left(a_i\right)}\left[\left(\rho + b - f^{'}\left(x\right)\right)U^{'}\left(a_i\right) - nD^{'}\left(x\right)\right],\ i = 1, ..., n \tag{2.8}$$

The system of differential equations (2.6) and (2.8) forms the MHDS in the state-control space. A steady state for this system characterizing the cooperative solution is defined as $\left(\bar{x}, \bar{a}\right) : \dot{x} = 0, \dot{a}_i = 0$.

Assume a utility function with a constant elasticity of marginal utility

$$-\theta = \frac{U^{''}\left(a_i\right)a_i}{U'\left(a_i\right)}$$

which is specialized to

$$U\left(a_i\right) = \frac{a_i^{1-\theta}}{1-\theta},\ 0 < \theta < 1 \tag{2.9}$$

$$U\left(a_i\right) = \log a_i\ ,\ \theta = 1 \tag{2.10}$$

In order to get a characterization of the cooperative benchmark that is independent of n, we take $\theta = 1$. After multiplying with n, the differential equation for the control variable (2.8) under symmetry (so that a_i is the same for all i) can be expressed in terms of total loading a:

$$\dot{a} = -\left[\left(\rho + b - f^{'}(x)\right) a - D^{'}(x) a^2\right] \tag{2.11}$$

Then the steady state $\left(\bar{x}, \bar{a}\right)$ in the state-control space is obtained as the solution of the system

$$a\,|_{\dot{x}=0} = bx - f(x), \; a = \sum_{i=1}^{n} a_i, \; a_i \text{ the same for all } i \tag{2.12}$$

$$a\,|_{\dot{a}=0} = \left[\frac{\left(\rho + b - f^{'}(x)\right)}{D'(x)}\right] \tag{2.13}$$

As shown in Brock and Starrett (2000), under the assumptions made on the $U(a_i)$, $f(x)$, and $D(x)$ functions, the system (2.7) and (2.5) (or equivalently (2.12) and (2.13)) has in general an odd number of steady states. Brock and Starrett also show that locally unstable steady states, with possibly complex eigenvalues, lie between two locally stable steady states. The first and the last steady state are locally stable. Furthermore, the locally stable steady states have the saddle-point property with a one-dimensional globally stable manifold.

The open-loop Nash equilibrium is determined by considering that each agent i behaves non-cooperatively. By taking the actions of the other agents $j \neq i$ as fixed, each agent maximizes the present value of its own net benefits (2.2) subject to (2.1). The current-value Hamiltonian is defined as

$$\mathcal{G}^i = U(a_i) - D(x) + \mu_i \left[a_i + \sum_{j \neq i}^{n} \bar{a}_j - bx + f(x)\right] \tag{2.14}$$

The necessary conditions of the maximum principle are

$$U'(a_i) = -\mu_i, \; i = 1, ..., n \tag{2.15}$$

$$\dot{\mu}_i = \rho\mu_i - \frac{\partial \mathcal{G}^i}{\partial x} = \left(\rho + b - f^{'}(x)\right)\mu_i + D^{'}(x), \; i = 1, ..., n \tag{2.16}$$

$$\dot{x} = a - bx + f(x), \; x(0) = x^0 \tag{2.17}$$

along with the transversality conditions at infinity,

$$\lim_{t\to\infty} e^{-\rho t}\mu_i(t) = 0, \lim_{t\to\infty} e^{-\rho t}\mu_i(t)\, x(t) = 0, \; i = 1, ..., n$$

Using (2.15) to eliminate μ_i from (2.16) as in the cooperative case and using the assumptions on the utility function, symmetry and multiplication by n yield

$$\dot{a} = -\left[\left(\rho + b - f^{'}(x)\right)a - \frac{1}{n}D^{'}(x)\, a^2\right] \tag{2.18}$$

The OLNE is characterized by (2.17) and (2.18). OLNE steady states are determined in the same way as in the cooperative case and they have the same properties in terms of the number of steady states and the stability properties. The steady states are solutions of the system

$$a\,|_{\dot{a}=0} = \left[\frac{n\left(\rho + b - f^{'}(x)\right)}{D'(x)}\right] \text{ and } (2.12) \tag{2.19}$$

Comparing the systems characterizing the cooperative and the OLNE steady states, that is comparing (2.19) with (2.13), it can be easily seen that the $a\,|_{\dot{a}=0}$ curve in the open-loop case shifts upwards relative to the $a\,|_{\dot{a}=0}$ curve in the cooperative case. This implies, for example, in the case of eutrophication of shallow lakes (see below) that the pollutant accumulation for the locally stable OLNE steady states is higher than the pollutant accumulation for the corresponding locally stable cooperative steady states, which is a well-known result for this model.

Having characterized the cooperative solution and the open-loop Nash equilibria, we now turn to the analysis of the feedback Nash equilibria.

3. Feedback Nash Equilibria

The feedback Nash equilibrium for our problem is defined as the solution of the maximization

$$\max_{\{a_i\}} J^i = \int_0^\infty e^{-\rho t} \left[U(a_i) - D\left(x\left(t\right)\right) \right] dt \text{ subject to (2.1)} \tag{3.1}$$

where the feedback Nash equilibrium strategy for player i is defined in a time-stationary feedback form as:

$$a_i = h_i\left(x\left(t\right)\right),\ \forall i \tag{3.2}$$

Suppose that $a_i = h_i^*\left(x\left(t\right)\right)$ provides a feedback Nash equilibrium (FBNE) such that $h_i^*\left(x\left(t\right)\right)$ is continuously differentiable. By fixing all players strategies except the ith one at $h_j = h_j^*$, $\forall j \neq i$, the problem for player i can be viewed as an optimal control problem with the current-value Hamiltonian for player i defined as

$$\mathcal{H}^i = U\left(a_i\right) - D\left(x\right) + \lambda_i \left[a_i + \sum_{j \neq i}^{n} h_j^*\left(x\right) - bx + f\left(x\right) \right] \tag{3.3}$$

The necessary conditions for optimality following from the application of Pontryagin's maximum principle are

$$U'(a_i) = -\lambda_i,\ i = 1, ..., n \tag{3.4}$$

$$\dot{\lambda}_i = \rho\lambda_i - \frac{\partial \mathcal{H}^i}{\partial x} = \left(\rho + b - \sum_{j\neq i}^{n} h_j^{*'}(x) - f'(x)\right)\lambda_i + D'(x) \tag{3.5}$$

$$\dot{x} = a_i + \sum_{j\neq i}^{n} h_j^{*}(x) - bx + f(x),\ x(0) = x^0 \tag{3.6}$$

along with the transversality conditions at infinity

$$\lim_{t\to\infty} e^{-\rho t}\lambda_i(t) = 0, \lim_{t\to\infty} e^{-\rho t}\lambda_i(t)\, x(t) = 0,\ i = 1, ..., n$$

From the conditions it is clear that the feedback Nash equilibrium is the same as the open-loop Nash equilibrium if $h_j^{*'}(x) = 0,\ \forall j$. Solving (3.4) for a_i we obtain $a_i = U'^{-1}(-\lambda_i)$. Substituting this solution into (3.6), we obtain for the state equation:

$$\dot{x} = U'^{-1}(-\lambda_i) + \sum_{j\neq i}^{n} h_j^{*}(x) - bx + f(x) \tag{3.7}$$

Equations (3.5) and (3.7) form the Modified Hamiltonian Dynamic System (MHDS) for the optimal control problem for player i. A steady state for this system, which is the steady state of the FBNE, is defined as $\left(\bar{x}, \bar{\lambda}_i\right) : \dot{x} = 0, \dot{\lambda}_i = 0$, with $\bar{a}_i = U'^{-1}\left(-\bar{\lambda}_i\right)$.

A MHDS and a FBNE steady state can be equivalently defined in the state-control space in analogy with the cooperative solution and the open-loop equilibrium above. Substituting (3.4) into (3.5) and using the assumptions on the utility function, a system of differential equations results that forms the MHDS in the state-control space for the feedback problem. Moreover, symmetry and multiplication by n yield

$$\dot{a} = -\left[\left(\rho + b - (n-1)\, h^{*'}(x) - f'(x)\right) a - \frac{1}{n} D'(x)\, a^2\right] \tag{3.8}$$

where $h^* = h_i^* = h_j^*, j \neq i$.

Again it is clear that the feedback equilibrium is the same as the open-loop equilibrium if $h^{*'}(x) = 0$, $\forall i$. Equation (3.8) is the counterpart of (2.11) for the cooperative solution if $n = 1$, and the counterpart of (2.18) for the OLNE if $h^{*'}(x) = 0$.

The state trajectory is obtained as the solution of the differential equation

$$\dot{x} = nh^*(x) - bx + f(x)\ ,\ a_i = h^*(x) \text{ the same for all } i \tag{3.9}$$

The steady state $\left(\bar{x}, \bar{a}\right)$ in the state-control space is obtained as the solution of the system

$$a\,|_{\dot{x}=0} = bx - f(x) \tag{3.10}$$

$$a\,|_{\dot{a}=0} = \left[\frac{n\left(\rho + b - (n-1)\, h^{*'}(x) - f'(x)\right)}{D'(x)}\right] \tag{3.11}$$

All the above results are based on the necessary conditions for optimality. To examine sufficient conditions in more detail we start by recalling that sufficiency in optimal control can be approached either through the Mangasarian sufficiency theorem (Mangasarian, 1966) or through a weaker condition than in Mangasarian's theorem, the Arrow sufficiency theorem (Arrow and Kurz, 1970).

Let $F_0(a_i, x) = U(a_i) - D(x)$ and $F_1(a_i, x) = a_i + \sum_{j \neq i}^{n} h_j^*(x) - bx + f(x)$. Then, the Mangasarian sufficiency theorem requires that $F_0(a_i, x)$ and $F_1(a_i, x)$ are concave and differentiable in (a_i, x) for all t and that $\lambda_i(t) \geq 0$ for all t.[10] To apply the Arrow sufficiency theorem define the maximized Hamiltonian

$$\mathcal{H}^{i0}(x, \lambda_i) = \max_{a_i} \mathcal{H}^i(a_i, x, \lambda_i)$$

If $\mathcal{H}^{i0}$ is a concave function of x for any given λ_i and the transversality conditions at infinity

$$\lim_{t\to\infty} e^{-\rho t}\lambda_i(t) \geq 0, \ \lim_{t\to\infty} e^{-\rho t}\lambda_i(t)\, x(t) = 0, \ i = 1, ..., n$$

are satisfied then the conditions of the maximum principle are also sufficient for $(a_i^*(t), x^*(t), \lambda_i^*(t))$ to be an optimal solution of the feedback problem. Arrow's sufficiency condition will be satisfied if Mangasarian's sufficiency condition is satisfied. For a standard optimal control problem, checking for sufficiency is relatively straightforward since the functions $F_0(a_i, x)$ and $F_1(a_i, x)$ are given data. In fact sufficiency is satisfied for trajectories converging to the locally stable steady states of the cooperative solution and open-loop equilibria.[11] This is, however, not necessarily true for a feedback problem since the feedback equilibrium strategies $\mathbf{h}(x) = (h_1(x), ..., h_n(x))$ are not known a priori but are determined endogenously as part of the solution. We suggest to search for feedback equilibrium strategies $\mathbf{h}(x) = (h_1(x), ..., h_n(x))$ that in combination with the functions representing the data for the problem ($U(a_i)$, $D(x)$, bx and $f(x)$) satisfy the necessary and sufficient conditions for optimality. Although we cannot rule out a priori existence of feedback equilibrium strategies that do not satisfy these sufficient conditions, in general these conditions will hold. This implies that they can be used to restrict the candidate

[10] See for example Takayama (1985, p. 660).

[11] If the second derivatives of the maximized Hamiltonians for the corresponding cooperative or open-loop problems, $H^0_{\lambda\lambda}$ and $-H^0_{xx}$, are positive semi-definite and the discount rate is small, then we have locally stable steady states of the cooperative solution or the OLNE (Brock and Malliaris, 1989, p.143) . In this case the maximized Hamiltonian is convex in λ and concave in x and the Arrow sufficiency theorem is satisfied.

functions for feedback equilibrium strategies and this will prove to be very useful in restricting the area of search of the algorithm we are about to present. This is summarized in the following assumption.

Assumption. The FBNE strategies $\mathbf{h}^*(x) = (h_1^*(x), ..., h_n^*(x))$ have the property that either the Mangasarian or Arrow sufficiency conditions for optimality for the respective optimal control problems are satisfied, i.e. that either the functions $F_0(a_i, x)$ and $F_1(a_i, x)$ are concave in (a_i, x) for all i and t and $\lambda_i(t) \geq 0$, or the maximized Hamiltonian $\mathcal{H}^{i0}(x, \lambda_i)$ is concave in x for any given λ_i.

As it is well known, the value function is central in defining the dynamic programming representation of the feedback problem. The dynamic programming equation for this problem with feedback equilibrium strategies $a_i = h_i(x(t))$ is given by

$$\rho V^i(x) = \max_{a_i} \left[U(a_i) - D(x) + V_x^i(x) \left[a_i + \sum_{j \neq i}^{n} h_j^*(x) - bx + f(x) \right] \right] \tag{3.12}$$

where $V^i(x)$ is the value function associated with the optimal control problem of maximizing J^i over a_i. We assume that the value function is twice differentiable. The right-hand side of the dynamic programming equation can be written as

$$\max_{a_i} \mathcal{H}^i\left(a_i, x, V_x^i\right) = \mathcal{H}^{i0}\left(x, V_x^i\right)$$

Therefore

$$\rho V^i(x) = \mathcal{H}^{i0}(x, \lambda_i), \text{ with } \lambda_i = V_x^i$$

Thus it follows from the discussion about sufficiency that if $\mathbf{h}^*(x)$ is such that the Mangasarian or the Arrow sufficiency conditions are satisfied, then $V^i(x)$ is concave in x with $V^i_{xx}(x) \leq 0$.[12] The following proposition can be stated.

Proposition 1. *Assume that a FBNE strategy* $\mathbf{h}^*(x)$ *exists such that the sufficiency conditions for the optimal control problem for player* i *are satisfied, then the corresponding value function for player* i *is concave.*

Therefore satisfaction of the necessary and sufficient conditions for a FBNE requires that a combination of this FBNE strategy with other functions, that are part of the data of the problem, satisfies certain concavity requirements. Satisfaction of these requirements implies concavity of the value function.

The same concavity requirements relate to the stability of the steady state $\left(\bar{x}, \bar{\lambda}_i\right)$ of the optimal control problem for player i. If the maximized Hamiltonian $\mathcal{H}^{i0}(x, \lambda_i)$ is concave in x and convex in λ_i,[13] then $-\mathcal{H}^{i0}_{xx}$ and $\mathcal{H}^{i0}_{\lambda_i\lambda_i}$ are non-negative and the steady state $\left(\bar{x}, \bar{\lambda}_i\right)$ is globally asymptotically stable (GAS) for all bounded solutions $x^*(t)$, $\lambda_i^*(t)$ (Brock and Malliaris, 1989, p. 143). This stability requirement puts a similar restriction on the FBNE strategies as the sufficiency requirement. The following proposition can be stated.

Proposition 2. *Assume that a FBNE strategy* $\mathbf{h}^*(x)$ *exists such that the maximized Hamiltonian* $\mathcal{H}^{i0}(x, \lambda_i)$ *is concave in* x*. Then a steady state* $\left(\bar{x}, \bar{\lambda}_i\right)$ *exists that is GAS for all bounded solutions* $x^*(t)$, $\lambda_i^*(t)$*. That is* $(x^*(t), \lambda_i^*(t)) \to \left(\bar{x}, \bar{\lambda}_i\right)$ *as* $t \to \infty$.

The conclusion of the analysis above is that considerations of sufficiency and stability give strong arguments to restrict the search for FBNE strategies to ones that induce a concave value

[12] See also Brock and Malliaris (1989, p. 139).

[13] Convexity in λ_i follows from the definition of the maximized Hamiltonian regardless of whether F_0 and F_1 are concave.

function. The concavity of the value function, which is implied by the concavity of the maximized Hamiltonian, can then be used to assign properties to the FBNE strategies that are useful in determining the equilibrium.

3.1. Determining the FBNE

Let $\mathcal{Z}$ be an open set which contains a locally stable open-loop steady state and assume that a continuous and differentiable FBNE exists in $\mathcal{Z}$. If under symmetry $h^{'}(x) = 0$ for $x \in \mathcal{Z}$, then the feedback steady state will be the same as the open-loop steady state while if $h^{'}(x) \neq 0$ for $x \in \mathcal{Z}$, then the $a\left|_{\dot{a}=0}\right.$ isocline shifts relative to the open-loop isocline for all $x \in \mathcal{Z}$ and the feedback steady state will be different from the open-loop steady state. To determine the feedback equilibrium we use the dynamic programming representation of the feedback problem.

From the dynamic programming (DP) equation (3.12) we obtain under symmetry with $V = V^i$ for all i

$$U^{'}(a_i) = -V_x(x) \Rightarrow a_i = g(V_x(x)) =: h(x) \tag{3.13}$$

$$\frac{da_i}{dx} = h^{'}(x) = g^{'}(V_x(x))\, V_{xx}(x) \tag{3.14}$$

Writing $V_x(x) = z$, so that $U^{'}(a_i) = -z$, and differentiating totally we obtain

$$g^{'}(z) = -\frac{1}{U''} > 0 \tag{3.15}$$

The following proposition is then immediately clear.

Proposition 3. *If the value function associated with the dynamic programming equation (3.12)* *is concave, then* $h'(x) \leq 0$. *Under strict concavity* $h'(x) < 0$.

Although the feedback strategy is not known at this point we restrict it, according to the discussion in the previous section, to the space of feedback strategies that satisfy sufficiency and stability. Then the value function is concave and the proposition above puts restrictions on the feedback equilibrium strategies. It follows in the MHDS representation that the $a\,|_{\dot{a}=0}$ isocline shifts upwards relative to the open-loop isocline and the feedback steady state will be on one side of the open-loop steady state.

Using the optimality condition (3.13) and symmetry, the DP equation becomes

$$\rho V(x) = \left[U(h(x)) - D(x) - U'(h(x))\,[nh(x) - bx + f(x)]\right] \tag{3.16}$$

Differentiating (3.16) with respect to x and using the optimality condition again, we obtain the following non-linear differential equation in $h(x)$:[14]

$$\begin{aligned} -\rho U'(h(x)) &= U'(h(x))\,h'(x) - D'(x) - \\ &\quad U''(h(x))\,h'(x)\,[nh(x) - bx + f(x)] - \\ &\quad U'(h(x))\left[nh'(x) - b + f'(x)\right] \end{aligned} \tag{3.17}$$

which can be written as

$$h'(x) = G(h(x), x) \tag{3.18}$$

This differential equation should also satisfy a boundary condition determined by the steady

[14] See also Benveniste and Scheinkman (1979) and Sargent (1987, p. 21).

state of the feedback equilibrium $\left(\bar{x}, \bar{a} = nh\left(\bar{x}\right)\right)$:

$$0 = nh\left(\bar{x}\right) - b\bar{x} + f\left(\bar{x}\right) \tag{3.19}$$

$$h\left(\bar{x}\right) = \frac{b\bar{x} - f\left(\bar{x}\right)}{n} \tag{3.20}$$

It follows from standard existence theorems for differential equations that a continuous differentiable solution $h(x)$ exists for (3.18) in a domain

$$\bar{x} - A \leq x \leq \bar{x} + A,\ h\left(\bar{x}\right) - B \leq h(x) \leq h\left(\bar{x}\right) + B$$

If a feedback steady state $\bar{x}$ is known then a solution $h(x)$ for the differential equation (3.18) can be obtained numerically. For this solution to be the feedback equilibrium for $x \in \mathcal{X} = \left\{x : \left|x - \bar{x}\right| < A\right\}, \mathcal{X} \subseteq \mathcal{Z}$ it should satisfy the following requirements: $h(x)$ should produce a stable steady-state equilibrium for the state equation and $h(x)$ should satisfy the optimality condition for the dynamic programming equation. More formally, the following conditions should be satisfied by the solution $h(x)$:

- The solution $h(x)$ should produce a stable equilibrium for the state equation at $\bar{x}$, or

$$nh\left(\bar{x}\right) - b\bar{x} + f\left(\bar{x}\right) = 0 \tag{3.21}$$

$$nh'\left(\bar{x}\right) - b + f'\left(\bar{x}\right) < 0 \tag{3.22}$$

- With the value function $V(x)$ obtained by substituting the solution $h(x)$ in (3.16), the optimality condition should be satisfied, or

$$U^{'}\left(h\left(x\right)\right)=-V_{x}\left(x\right) \tag{3.23}$$

Based on the discussion above, the search for the FBNE will be conducted on one side of the open-loop steady state where the value function is concave. The algorithm will verify whether concavity indeed holds. Furthermore, the algorithm will also verify that the feedback steady state that is found is also a steady state in the MHDS representation of the feedback problem.

Since the FBNE is defined in an open neighborhood of a locally stable OLNE steady state, there is no reason to expect (in the presence of multiple locally stable equilibria alternating with locally unstable ones) that a global continuous FBNE will exist. Thus we seek local FBNE strategies that solve the feedback problem in open neighborhoods of the locally stable OLNE steady states.

The major obstacle in calculating the solution $h\left(x\right)$ for the differential equation above and verifying whether it is a FBNE strategy is the fact that the steady state $\bar{x}$ is unknown. Therefore, we adopt an iterative approach on x in an open set $\mathcal{X}$ on one side of the open-loop steady state. The algorithm can be described as follows.

Let x^{OL} be an open-loop steady state. Since we restrict ourselves to feedback strategies that satisfy sufficiency conditions, such that the value function is concave in $\mathcal{X}$, it follows that $h^{'}\left(x\right)\leq 0$ and that $\bar{x}$ will be either to the left or to the right of x^{OL} depending on the slope of the $a\left|_{\dot{a}=0}\right.$ isocline in the neighborhood of the open-loop steady state. If the slope is positive, as it is in the neighborhood of locally stable steady states for the shallow lake problem (see below), then $\bar{x}>x^{OL}$.

1. Candidate feedback steady states that exceed the open-loop steady state are considered. For a selected candidate steady state x^F we determine the boundary condition (3.20) and then we solve numerically the differential equation (3.18) to obtain a solution $h\left(x; x^F\right)$.

2. Stability of the state equation around the steady state is checked with (3.22) for $\bar{x} = x^F$.

3. The value function is numerically determined from (3.16) as

$$\begin{aligned} V\left(x; x^F\right) &= \frac{1}{\rho}\left[U\left(h\left(x; x^F\right)\right) - D\left(x\right)\right. \\ &\left. -U^{'}\left(h\left(x; x^F\right)\right)\left[nh\left(x; x^F\right) - bx + f\left(x\right)\right]\right] \end{aligned} \tag{3.24}$$

   Then its slope $V_x\left(x; x^F\right)$ is calculated in the region $\mathcal{X}$. Concavity of the value function is verified.

4. The optimality condition (3.23) is checked for $h\left(x; x^F\right)$ and $V_x\left(x; x^F\right)$. More precisely: the difference $\mathcal{D} = U^{'}\left(h\left(x; x^F\right)\right) - \left[-V_x\left(x; x^F\right)\right]$, $x \in \mathcal{X}$ is determined and the indicator

$$MSD = \sum_{x \in \mathcal{X}} \mathcal{D}^2 \tag{3.25}$$

   is calculated. The feedback equilibrium strategy is the one that results from the steady state $x^F \in \mathcal{X}$, for which the indicator MSD is minimal. In our numerical approach $\mathcal{X}$ is the region of concavity of the value function.

5. It is verified that the resulting x^F is a steady state of the corresponding MHDS given by (3.5) and (3.7).

4. Feedback equilibrium strategies for the shallow lake problem

We consider now the determination of the feedback equilibrium for the shallow lake problem[15] by specializing as follows:

$$\begin{aligned} U(a_i) &= \log a_i \\ D(x) &= cx^2 \\ f(x) &= \frac{x^2}{1+x^2} \end{aligned}$$

where a_i denotes the phosphorus loadings into the lake, x the accumulated phosphorus in the lake, and $f(x)$ the internal (positive) feedbacks.

The optimality conditions and the corresponding differential equations of the MHDS for the cooperative solution, the OLNE and the FBNE are given by

Cooperative solution

$$\begin{aligned} \frac{1}{a_i} &= -\lambda, \ a_i \text{ the same for all } i \\ \dot{\lambda} &= \left(\rho + b - \frac{2x}{(1+x^2)^2}\right)\lambda + n2cx \\ \dot{x} &= -\frac{n}{\lambda} - bx + \frac{x^2}{1+x^2} \\ \dot{a} &= -\left(\rho + b - \frac{2x}{(1+x^2)^2}\right)a + 2cxa^2 \end{aligned}$$

[15] For a detailed description of this problem see Mäler, Xepapadeas and de Zeeuw (2000), Brock and Starrett (2000) or Brock, Carpenter and Hanson (1999).

OLNE

$$
\begin{aligned}
\frac{1}{a_i} &= -\mu_i,\ a_i \text{ the same for all } i \\
\dot{\mu}_i &= \left(\rho + b - \frac{2x}{(1+x^2)^2}\right)\mu_i + 2cx \\
\dot{x} &= -\frac{n}{\mu_i} - bx + \frac{x^2}{1+x^2} \\
\dot{a} &= -\left(\rho + b - \frac{2x}{(1+x^2)^2}\right)a + \frac{2cxa^2}{n}
\end{aligned}
$$

FBNE

$$
\begin{aligned}
\frac{1}{a_i} &= -\lambda_i,\ a_i \text{ the same for all } i \\
\dot{\lambda}_i &= \left(\rho + b - \sum_{j\neq i} h_j^{*\prime}(x) - \frac{2x}{(1+x^2)^2}\right)\lambda_i + 2cx \\
\dot{x} &= -\frac{1}{\lambda_i} + \sum_{j\neq i} h_j^{*}(x) - bx + \frac{x^2}{1+x^2} \\
\dot{a} &= -\left(\rho + b - (n-1)\,h^{*\prime}(x) - \frac{2x}{(1+x^2)^2}\right)a + \frac{2cxa^2}{n}
\end{aligned}
$$

The dynamic programming equation for the feedback problem becomes

$$
\begin{aligned}
\rho V(x) &= \left[\ln h(x) - cx^2 - \frac{1}{h(x)}\left[nh(x) - bx + f(x)\right]\right] \text{ or} \\
\rho V(x) &= \left[\ln h(x) - cx^2 - n + \frac{bx}{h(x)} - \frac{f(x)}{h(x)}\right]
\end{aligned}
$$

Differentiating with respect to x and using the optimality condition, we obtain

$$-\frac{\rho}{h(x)} = \left[\frac{h'(x)}{h(x)} - 2cx + \frac{bh(x) - bxh'(x)}{(h(x))^2} - \frac{f'(x)h(x) - f(x)h'(x)}{(h(x))^2}\right]$$

Multiplying by $(h(x))^2$, substituting $f(x)$ and rearranging, we obtain the differential equation for the FBNE strategy:

$$h'(x)\left(h(x) - bx + \frac{x^2}{1+x^2}\right) + h(x)\left((\rho+b) - 2cxh(x) - \frac{2x}{(1+x^2)^2}\right) = 0 \quad (4.1)$$

$$h\left(x^F\right) = \frac{1}{n}\left(bx^F - \frac{x^{F2}}{1+x^{F2}}\right) \quad (4.2)$$

We use the equations for the cooperative solution and the OLNE to get benchmarks and then we solve numerically for the FBNE strategies. The choice for the parameter values $\{b = 0.6, \rho = 0.01, c = 1\}$ is motivated as follows (see also Mäler, Xepapadeas and de Zeeuw, 2000). The value for b is one that may give occurrence of multiple solutions and hysteresis but outcomes are reversible. The value for c reflects the relative weight attached to ecological damage. The choice for c and for the discount rate ρ are such that the cooperative solution is in the area where the lake is in a so called oligotrophic state, which means that it is relatively clean. For the number of agents high enough ($n = 2$ with these parameter values), non-cooperative Nash equilibria occur in this area but also in the area where the lake is in a so called eutrophic state, which means that it is relatively polluted.

Cooperative solution

For the cooperative solution, which is obtained by setting $n = 1$, we determine the steady-state phosphorus and loading levels as $x^* = 0.3361$, $a^* = 0.1001$, respectively. The steady state is shown in figure 1. The cooperative solution is unique and saddle-point stable with eigenvalues $\{1.88707, -0.944399\}$..

[Figure 1]

Open-Loop Nash Equilibria

For the open-loop Nash equilibria with $n = 2$, we obtain the three steady states shown in the table below and in figure 2.

x_1^{OL}	x_2^{OL}	x_3^{OL}
0.3699	0.9678	1.5384
a_1^{OL}	a_2^{OL}	a_3^{OL}
0.1016	0.0970	0.2201

The first and the third steady states are saddle-point stable, whereas the second one is unstable. The corresponding eigenvalues are

$$\{2.13191, -1.15949\}, \{0.458044 \pm 0.347793i\}, \{0.949304, -0.277843\}$$

[Figure 2]

Feedback Nash Equilibria

We consider first an open set containing x_1^{OL} and try candidate feedback steady states to the right of x_1^{OL}. The minimum MSD is obtained for $x^F = 0.46$, with corresponding steady-state loading for each player $a_i^F = 0.0507$, $i = 1, 2$, and a total loading of $a^F = 0.1014$. The

FBNE strategy is obtained as the numerical solution of (4.1) with boundary condition (4.2) with $x^F = 0.46$.

The solution of the differential equation is shown in Figure 3.[16] It has a negative slope in the region $x \in (0.37, 0.89)$. The value function is shown in figure 4. As is to be expected, the value function is concave in the region where the FBNE strategy has a negative slope.

[Figure 3]

[Figure 4]

After substitution of the FBNE strategy $h(x)$, the state equation is shown in figure 5.

[Figure 5]

The slope of the state equation around the steady state $x^F = 0.46$ is negative, so that this steady state is stable. Note that another steady state exists outside the region of concavity of the value function at $\bar{x} = 0.25$ which is unstable.

Finally, we substitute $h(x)$ in the MHDS for the optimal control problem for player i. The steady state is at $\left(\bar{x} = 0.46, \bar{\lambda}_i = -19.73\right)$ with eigenvalues $(-184.211, 0.00237)$ implying local saddle-point stability. The steady-state loading for player i is $\bar{a}_i = 0.0507$. Thus, the solution $h(x)$ verifies also the steady-state and stability requirements for the MHDS representation of the feedback problem.

We can conclude that the solution $h(x)$, $x \in (0.37, 0.89)$ is a non-linear FBNE strategy that satisfies the necessary and sufficient conditions for optimality.

Second, we consider an open set containing x_3^{OL} and try candidate feedback steady states to the right of x_3^{OL}. Following exactly the same steps as above, the minimum MSD is obtained for

[16] The numerical solution of the non-linear differential equation (4.1) was obtained using Mathematica 4.0.

$x^F = 1.70$, with corresponding steady-state loading for each player $a_i^F = 0.1385$, $i = 1, 2$, and a total loading of $a^F = 0.277$. The FBNE strategy is shown in figure 6.

[Figure 6]

The FBNE strategy is defined for $x > 1.53$.[17] The FBNE strategy is negatively sloping and the value function is concave. Stability of the state equation holds around the steady state $x^F = 1.7$. The steady state of the MHDS is also verified at $\left(\bar{x} = 1.7, \bar{\lambda}_i = -7.22\right)$ with eigenvalues $(-342.588, 0.0179)$ implying local saddle-point stability. The steady-state loading for player i is $\bar{a}_i = 0.1385$. Thus, the solution $h(x)$ verifies also the steady-state and stability requirements for the MHDS representation of the feedback problem.

5. Concluding Remarks

In this paper we present a method to determine (non-linear) feedback equilibrium strategies for a class of non-linear-quadratic differential games that is typical in the analysis of problems in environmental and resource management.

By considering the sufficiency and stability requirements in the Hamiltonian representation of the differential game, we can restrict the candidate feedback equilibrium strategies, and by using the dynamic programming representation of the game, we can construct a simple iterative algorithm on the possible steady states of the system. The search for feedback equilibrium strategies, that in combination with the data functions of the problem satisfy necessary and sufficient conditions for optimality, implies that the value functions are concave. This in turn

[17] To the left of 1.53 a singularity is suspected and the numerical solution of the differential equation is not possible.

implies, for the class of problems we are considering, that the slope of the feedback strategies is negative. This result is used to restrict the area of search for the possible steady states of the equilibrium.

Existence of multiple steady states, due to the non-linearity of the problem, implies that the feedback equilibrium strategies are determined locally in the basin of attraction of the corresponding open-loop equilibrium. We show that when the sufficient conditions for the Hamiltonian representation of the differential game are satisfied, the FBNE steady state, for typical problems in environmental and resource management, is “worse” than the corresponding OLNE steady state and thus worse than the cooperative one, contrary to previous results in the literature.

We apply our algorithm to the shallow lake problem and determine the non-linear feedback equilibrium strategies in both the oligotrophic and the eutrophic area of the lake.

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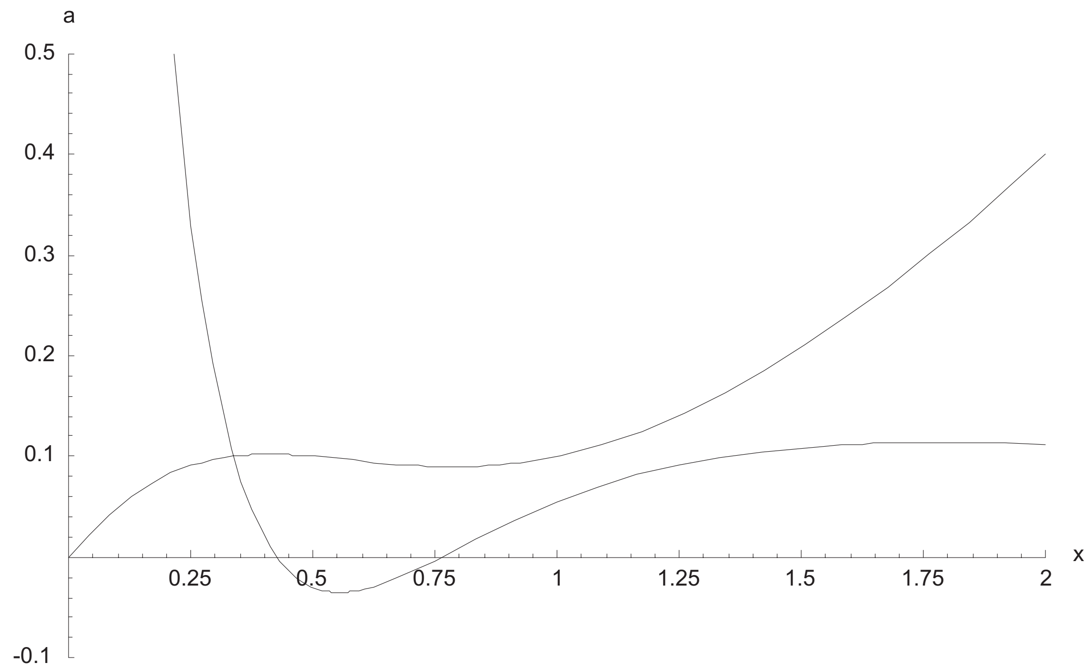

Figure 1: Cooperative solution for the shallow lake problem

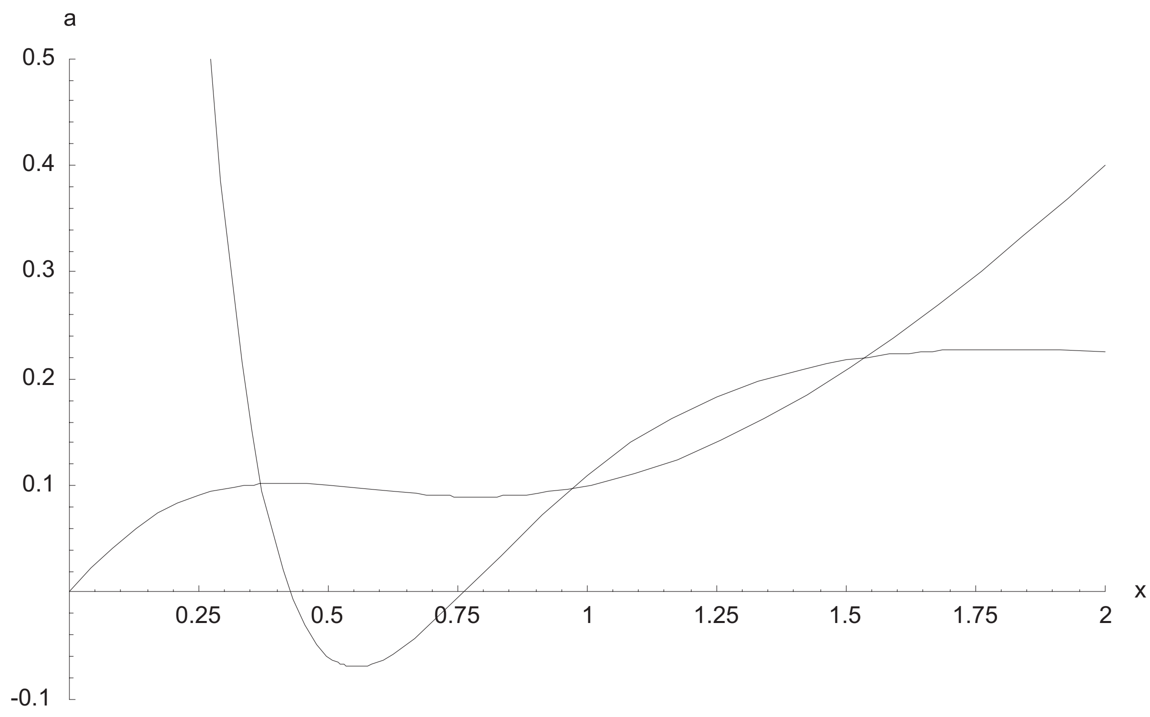

Figure 2: Open loop Nash equilibrium for the shallow lake problem

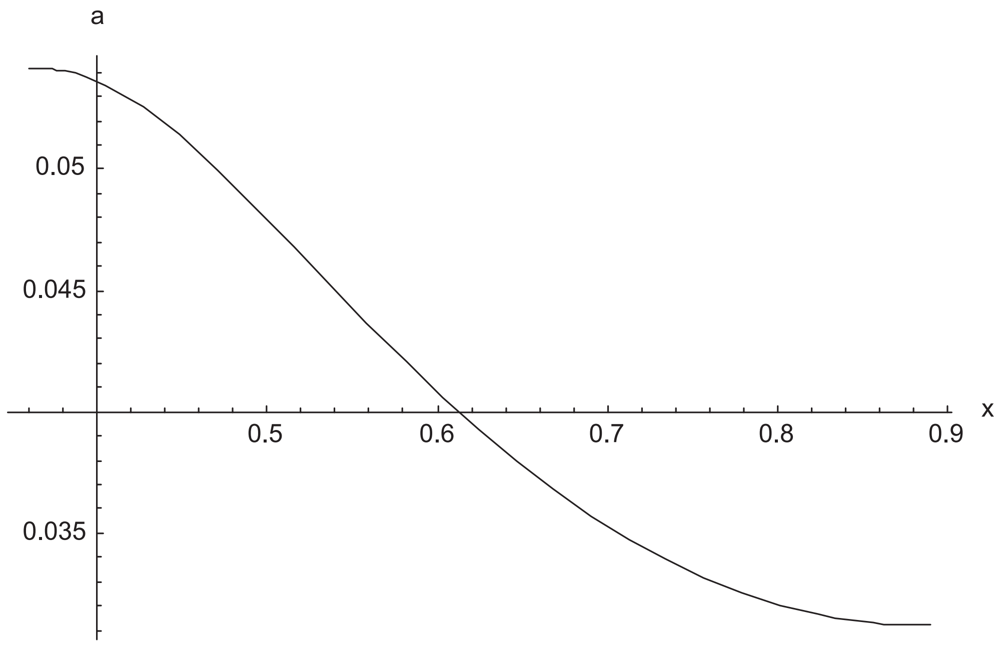

Figure 3: Feedback Nash equilibrium strategy in the neighborhood of x^{OL}_{1}

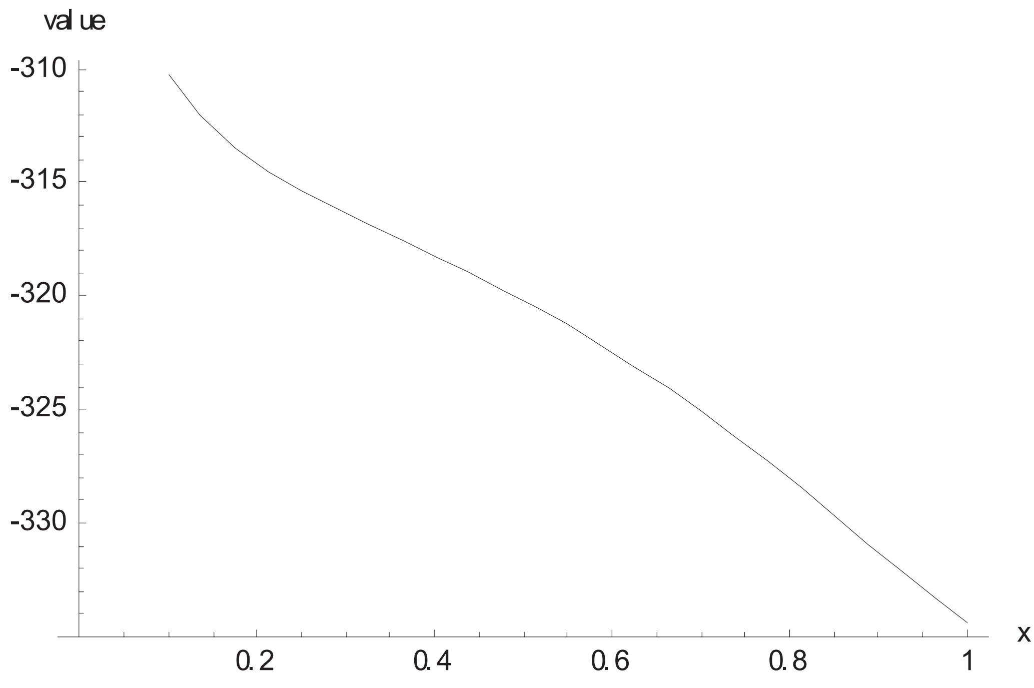

Figure 4: The Value function

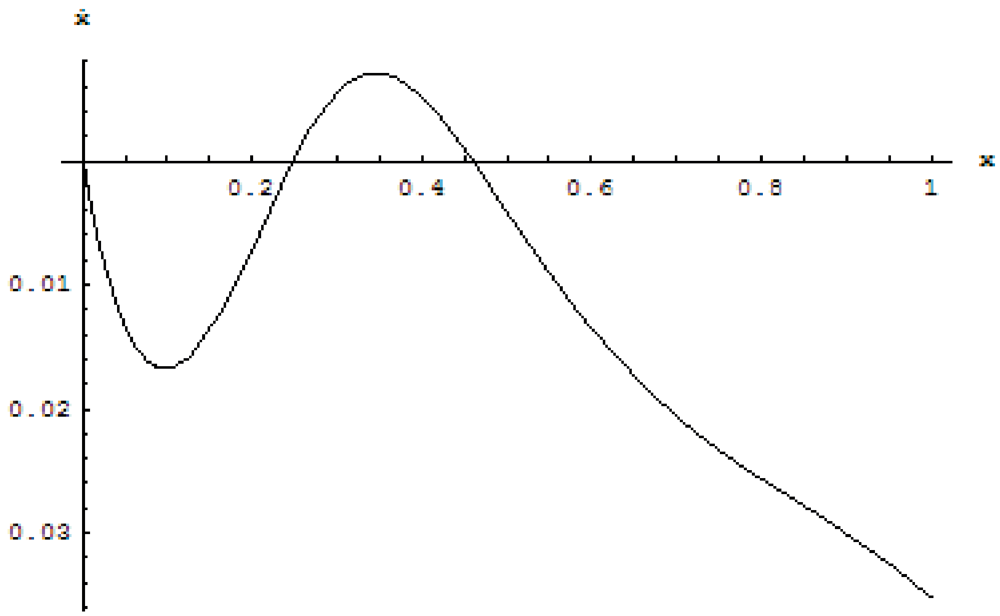

Figure 5. The state equation under the FBNE strategy

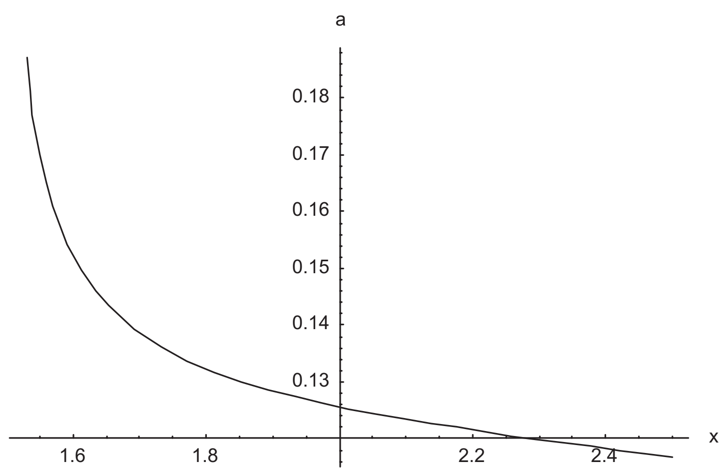

Figure 6: Feedback Nash equilibrium strategy in the neighborhood of x^{OL}_{3}